Abstract

This paper studies how foreign investors’ concerns about model misspecification affect sovereign bond spreads. We develop a general equilibrium model of sovereign debt with endogenous default wherein investors fear that the probability model of the underlying state of the borrowing economy is misspecified. Consequently, investors demand higher returns on their bond holdings to compensate for the default risk in the context of uncertainty. In contrast with the existing literature on sovereign default, we match the bond spreads dynamics observed in the data together with other business cycle features for Argentina, while preserving the default frequency at historical low levels.

Keywords: sovereign debt, default risk, model uncertainty, robust control.

JEL codes: D81, E21, E32, E43, F34.

1 Introduction

Sovereign defaults, or debt crises in general, are a pervasive economic phenomenon, especially among emerging economies. Recent defaults by Russia (1998), Ecuador (1999) and Argentina (2001), and the current debt crises of Greece have put sovereign default issues at the forefront.

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of economic policy discussion. Confronted with the risk of default (and further contingencies regarding the debt restructuring process), investors demand a compensation for bearing this risk, which, at the same time, hinders the access to credit of the borrowing economies. Thus, high and volatile bond spreads translate into high and volatile borrowing costs for these economies. Therefore, constructing economic models that can both generate these default events and provide accurate predictions in terms of pricing, is key.

As is the case in most of the asset pricing literature, the literature on defaultable debt follows the rational expectations paradigm; lenders fully trust the single probability model governing the state of the economy and are not concerned with any source of potential misspecification. It is well documented that economic models using this paradigm face difficulties when confronted with the asset prices data. The case of defaultable debt (either corporate or sovereign) is not an exception. In this case, models are typically unable to account for the observed dynamics in the bond spreads, while preserving the default frequency at historical levels.\(^1\) This paper tackles this “pricing puzzle”—while also accounting for other salient empirical features of the real economy—by studying how lenders’ desire to make decisions that are robust to model misspecification affects equilibrium prices and allocations.\(^2\)

In this paper we adapt the seminal general equilibrium model of sovereign default of Eaton and Gersovitz (1981) by introducing lenders that distrust their probability model governing the evolution of the state of the borrowing economy and want to guard themselves against misspecification errors in it. In the model, a borrower (e.g., an emerging economy) can trade long-term bonds with international lenders in financial markets. Debt repayments cannot be enforced and the emerging economy may decide to default. Lenders in equilibrium anticipate the default strategies of the emerging economies and demand higher returns on their sovereign bond holdings to compensate for the default risk. In case of default, the economy is temporarily excluded from financial markets and suffers a direct output cost. In this setting, we show how lenders’ desire to make decisions that are robust to misspecification of the conditional probability of the borrower’s endowment alters the returns on sovereign bond holdings.\(^3\)

The assumption about lenders’ concerns about model misspecification is intended to capture the fact that foreign lenders may distrust their statistical model used to predict relevant

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1. This phenomenon is not limited to the sovereign debt literature, since it is also well-documented in the corporate debt literature; see Huang and Huang (2003) and Elton et al. (2001), for example.
2. For a summary of the empirical regularities in emerging economies, see, e.g., Neumeyer and Perri (2005) and Uribe and Yue (2006).
3. By following Eaton and Gersovitz (1981), we abstract from transaction costs, liquidity restrictions, and other frictions that may affect the real return on sovereign bond holdings.
macroeconomic variables of the emerging economy (i.e., the borrower). In addition, foreign lenders are aware of the limited availability of reliable official data.\textsuperscript{4} This issue has become more severe in recent years in some emerging economies, particularly in Argentina, where the government’s intervention in the computation of the consumer price index is known worldwide, motivating warning calls for correction coming from international credit institutions. By under-reporting inflation, the Argentinean government has been over-reporting real GDP growth. Concerns about model misspecification can also be attributed to measurement errors, and lags in the release of the official statistics together with subsequent revisions. These arguments are aligned with the suggested view of putting econometricians and agents in a position with identical information, and limitations on their ability to estimate statistical models.

The novelty in our paper comes from the fact that lenders are uncertainty averse in the sense that they are unwilling or unable to fully trust a unique probability distribution or probability model for the endowment of the borrower, and at the same time dislike making decisions in the context of alternative probability models. To express these doubts about model uncertainty, following Hansen and Sargent (2005) we endow lenders with multiplier preferences.\textsuperscript{5,6} Lenders in our model share a reference or approximating probability model for the borrower’s endowment, which is their best estimate of the economic dynamics. They acknowledge, however, that it may be misspecified, and they express their doubts about model misspecification by contemplating alternative probability distributions that are statistical perturbations of the reference probability model. To make choices that perform well over this set of probability distributions, the lender acts as if contemplating a conditional worst-case probability that is distorted relative to his approximating one. This distorted distribution therefore arises fromperturbing the approximating model by slanting probability towards the states associated with low utility. In our model, these low-utility states for the

\textsuperscript{4}Boz et al. (2011) document the availability of significantly shorter time series for most relevant economic indicators in emerging economies than in developed ones. For example, in the database from the International Financial Statistics of the IMF the median length of available GDP time series at a quarterly frequency is 96 quarters in emerging economies, while in developed economies is 164.

\textsuperscript{5}Axiomatic foundations for this class of preferences have been provided by Strzalecki (2011).

\textsuperscript{6}In a wide class of environments, the utility recursion with multiplier preferences can be reinterpreted in terms of Epstein-Zin utility formulation, beyond the standard log period payoff specification. In such case, the typical probability distortion through which the agent’s uncertainty aversion is manifested would take the form of a risk-sensitive adjustment used to evaluate future streams of consumption. We view the contributions of model uncertainty and risk aversion not as mutually exclusive but rather as complementary, in line with Barillas et al. (2009) among others. In our framework, this apparent observational equivalence, however, does not apply, because the lender contemplates perturbations only to the probability model governing the evolution of the borrower’s endowment, and not to the probability distribution of reentry to financial markets, which is assumed to be fully trusted.
lender coincide with those in which the payoff of the sovereign bond is lower, because default occurs in the first place or the market value of the outstanding debt drops.

The main result of our paper is that by introducing lenders’ fears about model misspecification our calibration matches the high, volatile, and typically countercyclical bond spreads observed in the data for the Argentinean economy, together with standard business cycle features while keeping the default frequency at historical levels. At the same time, our model can account for the average risk-free rate observed in the data; model uncertainty in our economy does not alter the risk-free rate. Interestingly, we find that if the borrower can issue long-term debt model uncertainty almost does not affect quantitatively its level of indebtedness of the borrowing economy; the opposite is true with one-period bonds.

It is worth pointing out that in the simulations we also find that under plausible values of the parameters risk aversion alone on the lenders’ side with time-separable preferences is not sufficient to generate the observed risk premia; this is an analogous result to the equity premium puzzle studied in Mehra and Prescott (1985). Additionally, as the degree of lenders’ risk aversion increases, the average net risk-free rate declines, eventually to negative levels.

The intuition behind our results is as follows. Under the assumption that international lenders are risk neutral and have rational expectations (by fully trusting the data generating process), the equilibrium bond prices are simply given by the present value of adjusted conditional probabilities of not defaulting in future periods. Consequently, the pricing rule in these environments prescribes a strong connection between equilibrium prices and default probability. When calibrated to the data, matching the default frequency to historical levels (the consensus number for Argentina is around 3 percent annually), delivers spreads that are too low relative to those observed in the data. Our methodology breaks this strong connection by introducing a different probability measure, the one in which lenders’ uncertainty aversion is manifested. In our case, there is a strong connection between equilibrium prices and the default probability under this new worst-case probability measure. The probability distortion inherited in the worst-case density would induces in general a sufficiently negative correlation of the market stochastic discount factor with the payoff of the bond, which is the key element in generating high spreads while matching the default frequency.

Some recent papers, such as Arellano (2008), Arellano and Ramanarayanan (2012), and Hatchondo et al. (2010), assume instead an ad hoc functional form for the market stochastic discount factor. However, Arellano (2008), Borri and Verdelhan (2010), Lizarazo (2010), and Hatchondo et al. (2010), use a default frequency of 3 percent per year. Yue (2010) and Mendoza and Yue (2010) target an annual default frequency of 2.78 percent. Also, Reinhart et al. (2003) finds that emerging economies with at least one episode of external default or debt restructuring defaulted roughly speaking three times every 100 years over the period from 1824 to 1999.
discount factor in order to generate sizable bonds spreads as observed in the data. We show that for Gaussian processes for the borrower’s endowment these ad hoc functional forms are equivalent to a probability distortion that only shifts the conditional mean of the reference distribution. Our paper can therefore be seen as providing microfoundations for valid stochastic discount factors. We also find quantitative similarities in the pricing implications between these implied probability distortions and ours.

In our model with a defaultable asset, this endogenous probability distortion is discontinuous in the realization of the borrower’s next-period endowment as a result of the discontinuity in the payoff of the risky bond due to the default contingency. This yields an endogenous hump of the worst-case density over the interval of endowment realization in which default is optimal. This special feature is unique to this current setting. A direct implication of this is that the subjective probability assigns a significantly higher probability to the default event than the actual one. Since we can view the default event as a “disaster event” from the lenders’ perspective, this result links to the growing literature on “rare events”; see, for example, Barro (2006). Fears about model misspecification then amplify its effect on both allocations and equilibrium prices, as they increase the lenders’ perceived likelihood of these rare events occurring, leading to ”peso problems”. We find this an interesting contribution of our paper.

We also extend our benchmark model first by assuming a stochastic endowment for the lender, and second by letting lenders trade other financial assets beyond sovereign debt markets. For both cases, as long as lenders’ endowment and the payoff of these other financial assets are independent of the borrower’s endowment, the equilibrium bond prices or the borrower’s allocations remain unchanged. Besides the theoretical contribution, these results imply that there is no need to identify who the lenders are in the data, and, in particular, to find a good proxy of their income relatively to the borrower’s endowment. Moreover, when solving the model numerically, it is sufficient to keep track of the wealth of lenders only consisting of risky bonds.

Finally, in this paper we also present a methodological contribution that is of independent interest. The first methodological contribution of this paper relates to the way we solve the model numerically using the discrete state space (DSS) technique, in the context of model uncertainty. Since default is a discrete choice, it can occur that—under the DSS technique—the operator mapping prices to prices is not continuous, which may lead to convergence problems. We handle this technical complication by introducing an i.i.d. preference shock.8

8In their model of unsecured consumer credit and bankruptcy, Chatterjee et al. (2009) allow for a idiosyncratic preference shock to households. However, their preference shock technically differs from ours along
This preference shock enters additively in the autarky utility value of the borrower’s utility when it evaluates the default decision, and it is drawn from a logistic distribution, following McFadden (1981) and Rust (1994). As a result, the default decision, which was originally a discrete variable taking values of 0 or 1, becomes a continuous variable representing a probability that depends on the spread of borrower’s continuation values of repaying and defaulting on the outstanding debt. We show that, as the distribution of the preference shock converges to a point mass at zero (i.e. its variance converges to zero), if the equilibrium in the economy with the preference shock converges, it does so to the equilibrium in the economy without preference shock. It therefore implies that for sufficiently small preference shocks, the economy with the preference shock is closed to the original economy.

Roadmap. The paper is organized as follows. Section 2 presents the model. In Section 3 we describe the implications of model uncertainty on equilibrium prices. In Section 4 considers the extensions to our theoretical framework and derives equilibrium results. In Section 5 we calibrate our model to Argentinean data and present our quantitative results for long-term bonds. We also provide a comparison between one-period and long-term debt models, and conduct a robustness check. In Section 6 we relate our model to the papers using ad hoc functional forms for the stochastic discount factor. Section 7 disciplines the degree of robustness in our economy using detection error probabilities and provides an alternative interpretation of them. Finally, Section 8 concludes.

Related Literature. This paper builds and contributes to two main strands of the literature: sovereign default, and robust control theory and ambiguity aversion or Knightian uncertainty, in particular applied to asset pricing.

Arellano (2008) and Aguiar and Gopinath (2006) were the first to extend Eaton and Gersovitz (1981) general equilibrium framework with endogenous default and risk neutral lenders to study the business cycles of emerging economies. Chatterjee and Eyingungor several dimensions: it is a persistent, discrete shock that affects the per-period utility of the household in a multiplicative way regardless of its credit situation. Indeed, the authors motivate the introduction of this shock to capture the effects of marital disruptions, rather than to address convergence issues. In fact, to our knowledge, they do not provide theoretical convergence results.

We view our method as an alternative to the algorithm proposed by Chatterjee and Eyingungor (2012), based on an output shock. We think our methodology could be of independent interest and extended to other settings.

DEP measures the discrepancy between the approximating and the distorted models. Roughly speaking, DEP is akin to the type I error, which measures the probability of mistakenly rejecting the true model. Lenders in our economy are assumed to be concerned about models for the borrower’s endowment that are difficult to distinguish from one another given the available dataset.
(2012) introduced long-term debt in these environments. Lizarazo (2010) endows the lenders with constant relative risk aversion (CRRA) preferences. Borri and Verdelhan (2010) have studied the setup with positive correlation between lenders’ consumption and output in the emerging economy in addition to time-varying risk aversion on the lenders’ side as a result of habit formation.

From a technical perspective, Chatterjee and Eyingungor (2012) proposes an alternative approach to handle convergence issues. The authors consider an i.i.d. output shock drawn from a continuous distribution with a very small variance. Once this i.i.d. shock is incorporated, they are able to show the existence of a unique equilibrium price function for long-term debt with the property that the return on debt is increasing in the amount borrowed.

To our knowledge, the paper that is the closest to ours is the independent work by Costa (2009). That paper also assumes that foreign lenders want to guard themselves against specification errors in the stochastic process for the endowment of the borrower, but this is achieved in a different form. In our model, lenders are endowed with Hansen and Sargent (2005) multiplier preferences. With these preferences, lenders contemplate a set of alternative models and want to guard themselves against the model that minimizes their lifetime utility. In contrast, in Costa (2009) the worst-case density minimizes the expected value of the bond. Moreover, Costa (2009) considers one-period bonds and assumes lenders live for one period only.

Other recent studies that have focused on business cycles in emerging economies in the presence of fears about model misspecification are Young (2012) and Luo et al. (2012). Young (2012) studies optimal tax policies to deal with sudden stops when households and/or agents distrust the stochastic process for tradable total factor productivity shocks, trend productivity, and the interest rate. Luo et al. (2012) explores the role of robustness and information-processing constraints (rational inattention) in the joint dynamics of consumption, current account, and output in small open economies.

Finally, our paper relates to the growing literature analyzing the asset-pricing implications of ambiguity. Barillas et al. (2009) find that introducing concerns about robustness to model misspecification can yield combinations of the market price of risk and the risk-free rate that approach Hansen and Jagannathan (1991) bounds. Using a dynamic portfolio choice problem of a robust investor, Maenhout (2004) can explain high levels of the equity premium, as observed in the data. Drechsler (2012) replicates several salient features of the joint dynamics of equity returns, equity index option prices, the risk-free rate, and conditional variances, in the context of Knightian uncertainty. Hansen and Sargent (2010)

2 The Model

In our model an emerging economy interacts with a continuum of identical foreign lenders of measure 1. The emerging economy is populated by a representative, risk-averse household and a government.

The government in the emerging economy can trade a long-term bond with atomistic foreign lenders to smooth consumption and allocate it optimally over time. Throughout the paper we will refer to the emerging economy as the borrower. Debt contracts cannot be enforced and the borrower may decide to default at any point of time. In case the government defaults on its debt, it incurs two types of costs. First, it is temporarily excluded from financial markets. Second, it suffers a direct output loss.

While the borrower fully trusts the probability model governing the evolution of its endowment, which we will refer to as the approximating model, the lender suspects it is misspecified. From here on, we will use the terms probability model and distribution, interchangeably. For this reason, the lender contemplates a set of alternative models that are statistical perturbations of the approximating model, and wishes to design a decision rule that performs well across this set of distributions.

Throughout the paper, for a generic random variable $W$, we use $W$ to denote the random variable and $w$ to denote a particular realization.

\footnote{More asset-pricing applications with different formulations of ambiguity aversion are Epstein and Wang (1994), Chen and Epstein (2002), Hansen (2007) and Bidder and Smith (2011).}

\footnote{In order to depart as little as possible from Eaton and Gersovitz (1981) framework, throughout the paper we assume that the lender distrusts only the probability model dictating the evolution of the endowment of the borrower, not the distribution of any other source of uncertainty, such as the random variable that indicates whether the borrower re-enters financial markets or not. At the same time, and for the same reason, we assume the extreme case of no doubts about model misspecification on the borrower’s side.}
Time is discrete $t = 0, 1, \ldots$. Let $(W_t)_{t=0}^\infty \equiv (X_t, Y_t)_{t=0}^\infty$ be a stochastic process describing the borrower’s endowment. In particular, let $(Y_t)_{t=0}^\infty$ be a discrete-state Markov Chain, $(Y, P_{Y|Y}, \nu)$ where $Y \equiv \{y_1, \ldots, y_{|Y|}\} \subseteq \mathbb{R}_+$, $P_{Y|Y}$ is the transition matrix and $\nu$ is the initial probability measure, which is assumed to be the (unique) invariant (and ergodic) distribution of $P_{Y|Y}$. Let $(X_t)_{t=0}^\infty$ be such that, for all $t$, $X_t \in [x, \bar{x}] \equiv X \subseteq \mathbb{R}$ is an i.i.d. continuous random variable, i.e., $X_t \sim P_X$ and $P_X$ admits a pdf (with respect to Lebesgue), which we denote as $f_X$. Henceforth, we define $W \equiv X \times Y$ and $P_{W|W}$ denotes the conditional probability of $W_{t+1}$, given $W_t$, given by the product of $P_{Y|Y}$ and $P_X$; $P$ denotes the probability, induced by $P_{W|W}$ over infinite histories, $w^\infty = (w_0, \ldots, w_t, \ldots)$; finally, we also use $W^t$ to denote the $\sigma$-algebra generated by the partial history $W^t \equiv (W_0, W_1, \ldots, W_t)$.

The reason behind our definition of $W_t$ will become apparent below, but, essentially, we think of $Y_t + X_t$ as the borrower’s endowment at time $t$, and the separation between $Y_t$ and $X_t$ is due to numerical issues that appear in the method for solving the model; see Chatterjee and Eyingunog (2012) for a more thorough discussion.

Finally, we use $\bar{z}$ to denote the endowment of the lender, which is chosen to be non-random and constant over time for simplicity.

In what follows we adopt a recursive formulation for both the borrower and lender’s problem. We still use $t$ and $t+1$ to denote current and next period’s variables, respectively.

### 2.1 Timing Protocol

We assume that all economic agents, lenders, and the government (which cares about the consumption of the representative household), act sequentially, choosing their allocations period by period.

The economy can be in one of two stages at the beginning of each period $t$: financial autarky or with access to financial markets.

The timing protocol within each period is as follows. First, the endowments are realized. If the government has access to financial markets, it decides whether to repay its outstanding debt obligations or not. If it decides to repay, it chooses new bond holdings and how much to consume. Then, atomistic foreign lenders—taking prices as given—choose how much to save and how much to consume. The minimizing agent, who is a metaphor for the lenders’ fears about model misspecification, chooses the probability distortions to minimize the lenders’ expected utility. Due to the zero-sumness of the game between the lender and its minimizing agent, different timing protocols of their moves yield the same solution. If the government decides to default, it switches to autarky for a random number of periods.
While the government is excluded from financial markets, it has no decision to make and simply awaits re-entry to financial markets.

### 2.2 Sovereign Debt Markets

Financial markets are incomplete. Only a non-contingent, long-term bond can be traded between the borrower and the lenders. The borrower, however, can default on this bond at any time, thereby adding some degree of state contingency.

As in recent studies, the long-term bond exhibits a simplified payoff structure. We assume that in each period a fraction $\lambda$ of the bond matures, while a coupon $\psi$ is paid off for the remaining fraction $1 - \lambda$, which is carried over into next period. Modelling the bond this way is convenient to keep the problem tractable by avoiding too many state variables. Under these assumptions, it is sufficient to keep track of the outstanding quantity of bonds of the borrower to describe his financial position.

Bond holdings of the government and of the individual lenders, denoted by $B_t \in \mathbb{B} \subseteq \mathbb{R}$ and $b_t \in \mathbb{B} \subseteq \mathbb{R}$, respectively, are $W^{t-1}$-measurable. The set $\mathbb{B}$ is bounded and thereby includes possible borrowing or savings limits.

Positive bond holdings $B_t$ means that the government enters period $t$ with net savings, that is, in net term it has been purchasing bonds in the past. Negative bond holdings $B_t$ means that the government enters period $t$ with net debt, that is, it has been borrowing in the past by selling bonds.

The borrower can choose a new quantity of bonds $B_{t+1}$ at a price $q_t$. A debt contract is given by a vector $(B_{t+1}, q_t)$ of quantities of bonds and corresponding bond prices; $\psi$ and $\lambda$ are primitives in our model. The price $q_t$ depends on the borrower’s demand for debt at time $t$, $B_{t+1}$, and his endowment $y_t$, since these variables affect his incentives to default. In this class of models, generally, the higher the level of indebtedness and/or the lower the (persistent) borrower’s endowment, the greater the chances the borrower will default (in future periods) and, hence, the lower the bond prices in the current period.

For each $y \in \mathbb{Y}$, we refer to $q(y, \cdot) : \mathbb{B} \to \mathbb{R}_+$ as the bond price function.\(^{13}\) Thus, we can define the set of debt contracts available to the borrower for a given $w$ as the graph of $q(y, \cdot)$.\(^{14}\)

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\(^{13}\)As we show below, the bond price function only depends on $(y_t, B_{t+1})$, and not on $x_t$.

\(^{14}\)The graph of a function, $f : \mathbb{X} \to \mathbb{Y}$, is the set of $\{(x, y) \in \mathbb{X} \times \mathbb{Y} : y = f(x) \text{ and } x \in \mathbb{X}\}$. 
2.3 Borrower’s Preferences

A representative household in the emerging economy derives utility from consumption of a single good in the economy. Its preferences over consumption plans can be described by the expected lifetime utility\(^\text{15}\)

\[
E \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \mid w_0 \right],
\]

where \(E \left[ \cdot \mid w_0 \right]\) denotes the expectation under the probability measure \(P\) (conditional on time zero information \(w_0\)), \(\beta \in (0, 1)\) denotes the time discount factor, and the period utility function \(u : \mathbb{R}_+ \rightarrow \mathbb{R}\) is strictly increasing and strictly concave, and satisfies the Inada conditions.

Note that the assumption that the representative household and the government fully trust the approximating model \(P\) is embedded in \(E \left[ \cdot \mid w_0 \right]\).

The government in this economy, which is benevolent and maximizes the household’s utility (1), may have access to international financial markets, where it can trade a long-term bond with the foreign lenders. While the government has access to the financial markets, it can sell or purchase bonds from the lenders and make a lump-sum transfer across households to help them smooth consumption over time. Debt is also used to front load consumption, as the borrower is more impatient than the international lenders, i.e. \(\beta < \gamma\) (where \(\gamma\) is the discount factor for the representative lender).

2.4 Borrower’s Problem

For each \((w_t, B_t)\), let \(V(w_t, B_t)\) be the value (in terms of lifetime utility) for the borrower of having the option to default, given an endowment vector \(w_t\), and outstanding bond holdings equal to \(B_t\). Formally, the borrower’s value of having access to financial markets \(V(w_t, B_t)\) is given by

\[
V(w_t, B_t) = \max \{V_A(x_t, y_t), V_R(w_t, B_t)\},
\]

where \(V_A(x_t, y_t)\) is the value of exercising the option to default, given an endowment vector \(w_t = (y_t, x_t)\), and \(V_R(w_t, B_t)\) is the value of repaying the outstanding debt, given state \((w_t, B_t)\). In the period announcing default, the continuous component of endowment \(x_t\) drops to its lowest level \(x\). For the rest of the autarky periods, however, \(x_t\) is stochastic and drawn from the distribution \(P_X\), mentioned before. Throughout the paper we use subscripts \(A\) and \(R\) to denote the values for autarky and repayment, respectively.

\(^{15}\) A consumption plan is an stochastic process, \((c_t)_{t},\) such that \(c_t\) is \(\mathcal{F}_t\)-measurable.
Every period the government enters with access to financial markets, it evaluates the present lifetime utility of households if debt contracts are honored against the present lifetime utility of households if they are repudiated. If the former outweighs the latter, the government decides to comply with the contracts, makes the principal and coupon payments for the debt carried from the last period $B_t$, totaling $(\lambda + (1 - \lambda)\psi)B_t$, and chooses next period’s bond holdings $B_{t+1}$. Otherwise, if the utility of defaulting on the outstanding debt and switching to financial autarky is higher, the government decides to default on the sovereign debt.

Consequently, the value of repayment $V_R(w_t, B_t)$ is

$V_R(w_t, B_t) = \max_{B_{t+1} \in \mathbb{B}} u(c_t) + \beta E[V(W_{t+1}, B_{t+1}) | w_t]$  
$s.t. c_t = y_t + x_t - q(y_t, B_{t+1})(B_{t+1} - (1 - \lambda)B_t) + (\lambda + (1 - \lambda)\psi)B_t.$

Finally, the value of autarky $V_A(w_t)$ is

$V_A(w_t) = u(y_t + x_t - \phi(y_t)) + \beta E[(1 - \pi)V_A(W_{t+1}) + \pi V(W_{t+1}, 0) | w_t],$

where $\pi$ is the probability of re-entering financial markets next period. In that event, the borrower enters next period carrying no debt, $B_{t+1} = 0$. The function $\phi : \mathbb{Y} \to \mathbb{Y}$ such that $y \geq \phi(y) \ \forall y \in \mathbb{Y}$ represents an ad hoc direct output cost on $y_t$, in terms of consumption units, that the borrower incurs when excluded from financial markets. This output loss function is consistent with evidence that shows that countries experience a fall in output in times of default due to the lack of short-term trade credit. Notice that in autarky the borrower has no decision to make and simply consumes $y_t - \phi(y_t) + x_t$.

The default decisions are expressed by the indicator $\delta : \mathbb{W} \times \mathbb{B} \to \{0, 1\}$, that takes value

\textsuperscript{16}Henceforth, $E[\cdot | w]$ denotes the expectation under the conditional distribution associated to the approximating model, given $w$.

\textsuperscript{17}As in Arellano (2008), we do not model the exclusion from financial markets as an endogenous decision by the lenders. By modeling this punishment explicitly in long-term financial relationships, Kletzer and Wright (1993) show how international borrowing can be sustained in equilibrium through this single credible threat.

\textsuperscript{18}Notice that we assume there is no debt renegotiation nor any form of debt restructuring mechanism. Yue (2010) models a debt renegotiation process as a Nash bargaining game played by the borrower and lenders. For more examples of debt renegotiation, see Benjamin and Wright (2009) and Pitchford and Wright (2012). Pouzo (2010) assumes a debt restructuring mechanism in which the borrower receives random exogenous offers to repay a fraction of the defaulted debt. A positive rate of debt recovery gives rise to positive prices for defaulted debt that can be traded amongst lenders in secondary markets.

\textsuperscript{19}Mendoza and Yue (2010) endogenize this output loss as an outcome that results from the substitution of imported inputs by less-efficient domestic ones as credit lines are cut when the country declares a default.
0 if default is optimal; and 1, otherwise; i.e., for all \((x, y, B)\),

\[
\delta(x, y, B) = I\{V_R(x, y, B) \geq V_A(x, y)\}.
\]

2.5 Lenders’ Preferences and their Fears about Model Misspecification

We assume that the lenders’ have per-period payoff linear in consumption, while also being uncertainty averse or ambiguity averse.\(^{20}\) Since the i.i.d. component \(x_t\) is introduced merely for computational purposes—to guarantee convergence, as in Chatterjee and Eyingungor (2012)—, we assume no doubts about the specification of its distribution.

The lenders distrust, however, the probability model which dictates the evolution of \(y_t\), given by the approximating model \(P_{Y|Y}\). For this reason, they contemplate a set of alternative densities that are statistical perturbations of the approximating model, and they wish to design a decision rule that performs well over this set of priors. These alternative conditional probabilities, denoted by \(\tilde{P}_{Y,t}(\cdot|w^t)\) for all \((t, w^t)\), are assumed to be absolutely continuous with respect to \(P_{Y|Y}(\cdot|y_t)\), i.e. for all \(A \subseteq \mathbb{Y}\) and \(w^t \in \mathbb{W}^t\), if \(P_{Y|Y}(A|y_t) = 0\), then \(\tilde{P}_{Y,t}(A|w^t) = 0\).\(^{21}\)

In order to construct any of these distorted probabilities \(\tilde{P}_{Y,t}\), for each \(t\), let \(m_{t+1} : \mathbb{Y} \times \mathbb{W}^t \rightarrow \mathbb{R}_+\) be the conditional likelihood ratio, i.e., for any \(y_{t+1}\) and \(w^t\),

\[
m_{t+1}(y_{t+1}|w^t) = \begin{cases} \frac{\tilde{P}_{Y,t}(y_{t+1}|w^t)}{P_{Y|Y}(y_{t+1}|y_t)} & \text{if } P_{Y|Y}(y_{t+1}|y_t) > 0 \\ 1 & \text{if } P_{Y|Y}(y_{t+1}|y_t) = 0 \end{cases}.
\]

Observe that for any \((t, w^t)\), \(m_{t+1}(\cdot|w^t) \in \mathcal{M}\) where \(\mathcal{M} \equiv \{g : \mathbb{Y} \rightarrow \mathbb{R}_+ | \sum_{y' \in \mathbb{Y}} g(y')P_{Y|Y}(y'|y) = 1, \forall y \in \mathbb{Y}\}\).

Following Hansen and Sargent (2008) and references therein, to express fears about model misspecification we endow lenders with multiplier preferences. We can think of the lenders as playing a zero-sum game against a fictitious minimizing agent, who represents their doubts about model misspecification. While the lenders choose bond holdings to maximize their utility, the minimizing agent chooses a sequence of distorted conditional probabilities \((\tilde{P}_{Y,t+1})_t\),

\(^{20}\)The reason for this, is that we want to highlight the effects of uncertainty aversion on the prices, and other equilibrium quantities, in an otherwise standard dynamic general equilibrium model.

\(^{21}\)Note that the distorted probabilities \(\tilde{P}_{Y,t}\) do not necessarily inherit the properties of \(P_{Y|Y}\), such as its Markov structure. At the same time, they may depend on the history of past realizations of all shocks, including \(x^t\), as these may affect equilibrium allocations.
or equivalently a sequence of conditional likelihood ratios \((m_{t+1})_t\), to minimize it. The choice of probability distortions is not unconstrained but rather subject to a penalty cost.

The lenders’ preferences over consumption plans \(c^L\) after any history any \((t, w^t)\) can be represented by the following specification

\[
U_t(c^L; w^t) = c_t^L(w^t) + \gamma \min_{m_{t+1}(\cdot|w^t) \in \mathcal{M}} \left\{ E_Y \left[ m_{t+1}(Y_{t+1}|w^t)U_{t+1}(c^L; w^t, Y_{t+1}) \mid y_t \right] + \theta \mathcal{E}[m_{t+1}(\cdot|w^t)](y_t) \right\}, \tag{2}
\]

where \(\gamma \in (0,1)\) is the discount factor, the parameter \(\theta \in (\theta, +\infty]\) is a penalty parameter that measures the degree of concern about model misspecification\(^{22}\), and the mapping \(\mathcal{E} : \mathcal{M} \to L^\infty(\mathbb{Y})\) is the conditional relative entropy, defined as

\[
\mathcal{E}[\lambda](y) \equiv E_Y \left[ \lambda(Y') \log \lambda(Y') \mid y \right] \tag{3}
\]

for any \(\lambda \in \mathcal{M}\) and \(y \in \mathbb{Y}\). Finally, \(U_{t+1}(c^L; w^t, y_{t+1})\) is the expected value of \(U_t(c^L; w^t, y_{t+1}, X_{t+1})\), conditioned on \(y_{t+1}\), but before the realization of \(X_{t+1}\), i.e.

\[
U_{t+1}(c^L; w^t, y_{t+1}) \equiv E_X \left[ U_{t+1}(c^L; w^t, y_{t+1}, X_{t+1}) \right], \tag{4}
\]

and \(U_t(c^L; w^t)\) is the present value expected utility at time \(t\), given that the previous history is given by \(w^t\) and the agent follows a consumption plan \(c^L\). By looking at expressions (2) and (4) we see that the probability distortion \(m_{t+1}\) pre-multiplies the expected continuation value before the realization of \(X_{t+1}\), i.e. \(U_{t+1}(c^L; w^t, y_{t+1})\), in line with our measurability assumption. For the sequential formulation of the lenders’ lifetime utility and the derivation of the recursion (2)-(4), see Appendix D.

For any given history \(w^t\), \(\mathcal{E}[m_{t+1}(\cdot|w^t)](y_t)\) measures the discrepancy of the distorted conditional probability, \(\tilde{P}_{Y|t}(\cdot \mid w^t)\), with respect to the approximating conditional probability \(P_{Y|t}(\cdot \mid y_t)\). Through this entropy term, the minimizing agent is penalized whenever she chooses distorted probabilities that differ from the approximating model. The higher the value of \(\theta\), the more the minimizing agent is penalized. In the extreme case of \(\theta = +\infty\), there are no concerns about model misspecification and we are back to the standard environment where both borrower and lenders share the same model, given by \(P_{Y|Y}\).

The minimization problem conveys the ambiguity aversion. It is easy to see that it yields

\(^{22}\)The lower bound \(\theta\) is a breakdown value below which the minimization problem is not well-behaved; see Hansen and Sargent (2008) for details.
the following specification for \( m_{t+1} \) for all \( y_{t+1} \) and \( w^t \),

\[
m^*_t(y_{t+1}|w^t) = \frac{\exp \left\{ -U_{t+1}(c_{w^t}; w_{t+1}) \theta \right\}}{E_{Y^t} \left[ \exp \left\{ -U_{t+1}(c_{w^t}; w_{t+1}) \theta \right\} | y_t \right]},
\]

(5)

Note that the higher the continuation value \( U_{t+1}(c_{w^t}; w_{t+1}) \), the lower the probability distortion associated to it. That means that through her choice of \( m^*_t \), the minimizing agent pessimistically twists the conditional distribution \( P_{Y^t|Y} \) by putting more weight on continuation outcomes associated with lower utility for the lenders.

2.6 Lenders’ Problem

As it will become clear below, for the recursive equilibrium in our particular environment, the lifetime utility in the previous section becomes \( W_R(w_t, B_t, b_t) \) or \( W_A(y_t) \).\(^{23}\) Here, \( W_R(w_t, B_t, b_t) \) is the equilibrium value (in lifetime utility) of an individual lender with access to financial markets, given the state of the economy \((w_t, B_t, b_t)\). \( W_A(y_t) \) is analogously defined, but when the borrowing economy has no access to financial markets.

Since lenders are atomistic, each individual lender takes as given the aggregate debt \( B_t \).

24 When lender and borrower can engage in a new financial relationship, the lender’s min-max problem at state \((w_t, B_t, b_t)\), is given by:

\[
W_R(w_t, B_t, b_t) = \min_{m^*_t \in M} \max \left\{ c^L_t + \theta \mathcal{E}[m_R](y_t) + \gamma E_{Y^t} [m_R(Y_{t+1})W(Y_{t+1}, X_{t+1}, B_{t+1}, b_{t+1})|y_t] \right\}
\]

\[
s.t. \ c^L_t = \mathbb{T} + q(y_t, B_{t+1})(b_{t+1} - (1 - \lambda)b_t) - (\lambda + (1 - \lambda)\psi)b_t
\]

\[
B_{t+1} = \Gamma(w_t, B_t),
\]

where for all \( y_{t+1} \in \mathbb{Y} \) the continuation value \( W(y_{t+1}, B_{t+1}, b_{t+1}) \) is given by:

\[
W(y_{t+1}, B_{t+1}, b_{t+1}) = E_{X^t} \left[ W(y_{t+1}, X_{t+1}, B_{t+1}, b_{t+1}) \right],
\]

where \( W(w_{t+1}, B_{t+1}, b_{t+1}) \equiv \delta(w_{t+1}, B_{t+1})W_R(w_{t+1}, B_{t+1}, b_{t+1}) + (1 - \delta(w_{t+1}, B_{t+1}))W_A(y_{t+1}) \) is the value of the lender when the borrower is given the option to default at state \((w_{t+1}, B_{t+1}, b_{t+1})\),

\(^{23}\) As we will see, \( x_t \) is not a state variable for the lender’s problem in financial autarky due to its i.i.d. nature, and the fact that there is no decision making during that stage.

\(^{24}\) Remember that we denote \( b_t \) as the individual lender’s debt, while \( B_t \) refers to the representative lender’s debt.
and $\Gamma : \mathbb{W} \times \mathbb{B} \to \mathbb{B}$ is the perceived law of motion of the individual lender for the debt holdings of the borrower, $B_{t+1}$. Observe that the optimal choice of $m_R$, is a mapping from $(w_t, B_t, b_t) \in \mathbb{W} \times \mathbb{B}^2$ to $\mathcal{M}$.

A few remarks are in order regarding equation (6). First, lenders receive every period a non-stochastic endowment given by $z$. Since the per-period utility is linear in consumption, the level of $z$ does not affect the equilibrium bond prices, bond holdings, and default strategies in our original economy; see Subsection 4. Moreover, we show that by allowing for an independent stochastic endowment for lenders, fully trusted or not by them, does not affect neither the equilibrium borrower’s allocations nor prices. Second, in the current setup, besides the risky bonds, lenders are allowed to only trade a zero net supply risk-less claim to one unit of consumption next period. Since all lenders are identical, no trade in such a claim takes place in equilibrium. Introducing this riskless claim is, however, useful to determine a risk-free rate $r^f_t$—which, in equilibrium, is a non-stochastic and given by $1 + r^f_t = 1/\gamma$— and thereby compute bond spreads. In Subsection 4, we show we can allow for trading in a broader class of assets, not necessarily in zero net supply, without altering the equilibrium bond prices, bond holdings, and default strategies in our original economy.

In financial autarky, as with the borrower, the lender has no decision to make. The lender’s autarky value at state $(y_t)$, is thus given by

$$W_A(y_t) = \min_{m_A \in \mathcal{M}} \{ z + \theta \gamma \mathbb{E}[m_A(y_t)] + \gamma \mathbb{E} [m_A(Y_{t+1}) ((1 - \pi)W_A(Y_{t+1}) + \pi W(Y_{t+1}, 0, 0)) | y_t] \},$$

where $\pi$ is the re-entry probability to financial markets. Note that the optimal choice, $m_A$, is a mapping from $\mathcal{Y}$ to $\mathcal{M}$.

In contrast with the borrower’s case, no output loss is assumed for the lender during financial autarky.

2.7 Recursive Equilibrium

As is standard in the quantitative sovereign default models, we are interested in a recursive equilibrium in which all agents choose sequentially.

**Definition 2.1.** A collection of policy functions $\{c, c^L, B, b, m_R, m_A, \delta\}$ is given by mappings for consumption $c : \mathbb{W} \times \mathbb{B} \to \mathbb{R}_+$ and $c^L : \mathbb{W} \times \mathbb{B}^2 \to \mathbb{R}_+$, bond holdings $B : \mathbb{W} \times \mathbb{B} \to \mathbb{B}$ and $b : \mathbb{W} \times \mathbb{B}^2 \to \mathbb{B}$ for borrower and individual lender, respectively; and, probability distortions $m_R : \mathbb{W} \times \mathbb{B} \to \mathcal{M}$, $m_A : \mathcal{Y} \to \mathcal{M}$ and default decisions, $\delta : \mathbb{W} \times \mathbb{B} \to \{0, 1\}$.

**Definition 2.2.** A collection of value functions $\{V_R, V_A, W_R, W_A\}$ is given by mappings $V_R : \mathbb{W} \times \mathbb{B} \to \mathbb{R}$, $V_A : \mathbb{W} \to \mathbb{R}$, $W_R : \mathbb{W} \times \mathbb{B}^2 \to \mathbb{R}$, $W_A : \mathcal{Y} \to \mathbb{R}$. 

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Definition 2.3. A recursive equilibrium for our economy is a collection of policy functions \( \{c^*, c_{L^*}, B^*, b^*, m_R^*, m_A^*, \delta^* \} \), a collection of value functions \( \{V_R^*, V_A^*, W_R^*, W_A^* \} \), a perceived law of motion for the borrower’s bond holdings, and a price schedule such that:

1. Given perceived laws of motion for the debt and price schedule, policy functions, probability distortions, and value functions solve the borrower and individual lender’s optimization problems.

2. For all \((w, B) \in W \times B\), bond prices \(q(y, B^*(w, B))\) clear the financial markets, i.e.,
   \[ B^*(w, B) = b^*(w, B), \]

3. The actual and perceived laws of motion for debt holdings coincide, i.e., \(B^*(w, B) = \Gamma(w, B)\), for all \((w, B) \in W \times B\).

In our economy, lenders behave competitively. That is, they take bond prices \(q(y, B^*)\), and government debt as given, and optimally choose bond holdings and consumption. In contrast, when making its default and debt decisions, the government in the borrowing economy takes as given the equilibrium bond price function \(q(y, \cdot)\) and internalizes the fact that through its debt choice it affects its incentives to default in the future and hence current equilibrium bond prices. Finally, the market clears at the equilibrium price.

After imposing the market clearing condition given by point 3 above, vector \((w_t, B_t)\) is sufficient to describe the state variables for any agent in this economy. Hence, from here on, we consider \((w_t, B_t)\) as the state vector, common to the borrower and the individual lenders.

3 Equilibrium Bond Prices and Probability Distortions

In our competitive sovereign debt market, uncertainty-averse lenders make zero profits in expectation given their own beliefs. Hence, for an endowment level \(y_t\) and a loan size \(B_{t+1}\), the bond price function satisfies for all \((y_t, B_{t+1})\),

\[
q(y_t, B_{t+1}) = \gamma E_Y \left[ E_X \left[ \left( \lambda + (1 - \lambda)(\psi + q(Y_{t+1}, B^*(W_{t+1}, B_{t+1}))) \right) \delta^*(W_{t+1}, B_{t+1}) \right] m^*(Y_{t+1}; y_t, B_{t+1}) \middle| y_t \right],
\]

where \(m^*: \mathbb{Y}^2 \times B \rightarrow \mathbb{R}_+\) is given by

\[
m^*(y_{t+1}; y_t, B_{t+1}) \equiv \frac{\exp\left\{ -\frac{W^*(y_{t+1}, B_{t+1}, B_{t+1})}{\theta} \right\}}{E_Y \left[ \exp\left\{ -\frac{W^*(Y_{t+1}, B_{t+1}, B_{t+1})}{\theta} \right\} \middle| y_t \right]}.
\]
The function $m^*$ is essentially the reaction function for the probability distortions, which is consistent with the FOCs in the minimization problem (6) and the market clearing condition for debt.\(^{25}\)

Given the state of the economy next period, if defaults occurs, the payoff of the bond is zero. Otherwise, a fraction $\lambda$ of the bond matures while the remaining $(1 - \lambda)$ pays off a coupon $\psi$ and keeps a market value of $q(Y_{t+1}, B^*(W_{t+1}, B_{t+1}))$.

In equilibrium, for each state of the economy $(w_t, B_t)$ only one debt contract is traded between the borrower and the lenders and, hence, we observe a particular quantity of new bond holdings, $B^*(w_t, B_t)$, with an associated price $q(Y_{t+1}, B^*(W_{t+1}, B_{t+1}))$.\(^{26}\)

In the absence of fears about model uncertainty, i.e. $\theta = +\infty$, the probability distortion vanishes, i.e. $m^* = 1$, that means that lenders' beliefs coincide with the approximating distribution $P_{Y|\cdot|Y}$, and hence the price function (7) is the same as in the rational expectations environment of Chatterjee and Eyingunor (2012).

Under model uncertainty, the lender in this economy distrusts the conditional probability $P_{Y|\cdot|Y}$ and wants to guard himself against a worst-case distorted distribution for $y_{t+1}$, given by $m^*(\cdot; w_t, B^*(y_t, B_t))P_{Y|\cdot|Y}(\cdot|y_t)$. The fictitious minimizing agent, who represents its doubts about model misspecification, will be selecting this worst-case density by slanting probabilities towards the states associated with low continuation utility for the lender, as observed from the twisting formula given by equation (5). In the presence of default risk, the states associated with lower utility coincide with the states in which the borrower defaults and therefore the lender receives no repayment; naturally, those are the states assigned the highest probability distortions. In addition, in this economy with long-term debt, upon repayment, the payoff responds to variations in the next-period bond price. Hence, states in which the latter is lower will be associated with relatively higher probability distortions.

Figure 1 illustrates the optimal distorting of the probability of next period realization of $Y_{t+1}$, given current state $(w_t, B_t)$ with access to financial markets.\(^{27}\) $B_{t+1}$ is computed using the optimal debt policy, i.e. $B_{t+1} = B^*(w_t, B_t)$. In the top panel of this figure we plot the conditional approximating density and the distorted density for $y_{t+1}$, as well as its corresponding

---

\(^{25}\)Observe that, by construction, $m^*(y_{t+1}; y_t, B^*(w_t, B_t)) = m^*_R(y_{t+1}; w_t, B_t)$. While $m^*_R$ are the optimal probability distortions along the equilibrium path, commonly computed in the robust control literature for atomistic agents, the reaction function $m^*$ is a necessary object of interest in this environment to evaluate alternative debt choices for the borrower.

\(^{26}\)In Arellano (2008) competitive risk-neutral lenders are indifferent between any individual debt holdings level $b_{t+1}$. In our environment this is not true anymore. Taking $q(Y_{t+1}, B^*(W_{t+1}, B_{t+1}))$ and the borrower’s strategies as given, lenders solve a convex optimization problem with a strictly concave objective function and hence there is a unique interior solution for individual debt holdings.

\(^{27}\)For illustrative purposes, a low endowment $y_t$ and low bond holdings $B_t$, or equivalently high debt level, were suitably chosen to have considerable default risk under the approximating density. The current endowment level $y_t$ corresponds to half a standard deviation below its unconditional mean, and the bond holdings $B_t$ are set to the median of its unconditional distribution in the simulations. Also, current $x_t$ was set to zero.
probability distortion $m_{R}^{*}$. The shaded area corresponds to the range of values for the realization of $Y_{t+1}$ in which the borrower defaults with probability equal or higher than 50 percent (note that the default decision at $t+1$ also depends on the realization of $X_{t+1}$). The bottom panel plot depicts the expected payoff of the bond at $t+1$, before the realization of $X_{t+1}$, that is, $E_{X} \left[ \left( \lambda + (1 - \lambda) (\psi + q(Y_{t+1}, B^{*}(W_{t+1}, B_{t+1}))) \right) \delta^{*}(W_{t+1}, B_{t+1}) \right]$.

In order to minimize lenders’ expected utility, the minimizing agent places a discontinuous probability distortion $m_{R}^{*}(\cdot; w_{t}, B_{t})$ over next period realizations of $y_{t+1}$, with values strictly larger than 1 over the default interval, and strictly smaller than 1 where repayment is optimal. By doing so, the minimizing agent takes away probability mass from those states in which the borrower does not default, and puts it in turn on those low realizations of $y_{t+1}$ in which default is optimal for the borrower. For this particular state vector $(y_{t}, B_{t})$ in consideration, the conditional default probability under the approximating model is 9.3 percent quarterly, while under the distorted one it is 16.2 percent, almost twice as high. The discontinuity of $m_{R}^{*}$ follows from the discontinuity of the lenders’ utility value with respect to $y_{t+1}$, which in turn is due to the discontinuity in next-period payoff of the bond as function of $y_{t+1}$. Discrete jumps in the payoff structure are therefore key to generating a discontinuous stochastic discount factor in our environment.

![Approximating and distorted densities](image)

**Figure 1**: Approximating and distorted densities.

With long-term debt, additional probability distorting takes place over the repayment interval. Since the payoff of the bond remains state contingent due to its dependence on the next-period bond
price, so will the lenders’ utility. Consequently, states associated with relatively lower next-period prices will be assigned relatively higher weights.

The tilting of the probabilities by the minimizing agent generates an endogenous hump of the distorted density over the interval of $y_{t+1}$ associated with default risk, as observed in Figure 1. The bi-peaked form of the resulting distorted conditional density is non-standard in the robust control literature, in which it typically displays only a shift in the conditional mean from the approximating one.\textsuperscript{28} Table 7 in Appendix A reports distortions in several moments of $y_t$ for our economy.

In the parameterized version of the model, incentives are such that default typically occurs in low output states. Therefore, by assigning relatively higher probability to these states, lenders hold pessimistic beliefs about the evolution of the borrowing economy. Nevertheless, it is worth noting that if the borrower could default sometimes in high output states, in line with the findings by Tomz and Wright (2007), lenders’ beliefs in equilibrium would turn to be over optimistic.

Sovereign default events in our model can be interpreted as “disaster events”, which, in our economy, emerge endogenously from the borrower’s decision making and the lack of enforceability of debt contracts. Fears about model misspecification in turn amplify their effect on both allocations and equilibrium prices, as they increase the perceived likelihood in the mind of the lenders of these rare events occurring. As a result, the model can be viewed as generating endogenously varying disaster risk.

**State-dependency of probability distortions.** In our economy, probability distortions are state-dependent and thereby typically time-varying. The default risk under the approximating density and the quantity of bonds carried over to next period, which the borrower can default on, affect the extent to which the minimizing agent distorts lenders’ beliefs. Figure 2 shows the approximating and distorted density of next period $y_{t+1}$ for different combinations of current endowment and bond holdings, $(y_t, B_t)$.\textsuperscript{29}

By comparing the two panels in the top row (or the bottom row), we can see how the probability distortion changes with the level of current debt. In this general equilibrium framework, we need to take into account the optimal debt response of the borrower for the current state of the economy. For the state vectors in consideration, the higher the current level of indebtedness $B_t$, the more debt the borrower optimally chooses to carry into next period, $B_{t+1}$. To see how the perceived probability of default next period varies, we check at how the default risk under the approximating model and the probability distortions change as current bond holdings $B_t$ increase. First, the interval of realizations of $y_{t+1}$ for which the borrower defaults is enlarged. The larger the quantity of bonds that the borrower has to repay at $t + 1$, the greater the incentives it would have to not do

\textsuperscript{28}See, for example, Barillas et al. (2009) and Anderson et al. (2003).

\textsuperscript{29}Low and high endowment $y_{t+1}$ correspond to half and a quarter a standard deviation below the unconditional mean of $y_t$, respectively. The i.i.d. output shock $x_t$ is set again to zero. Also, low debt is given by the median of the debt unconditional distribution, and high debt corresponds to the 60th percentile.
it. Consequently, the default risk under the approximating model is higher. Also, those new states on which there is default with higher debt become now low-utility states for the lender, and hence probability distortions $m^*_R$ larger than 1 are assigned to them in the new, worst-case density.

Second, the change in levels of the probability distortions may not be straightforward. On the one hand, since for these cases, more bond holdings $B_{t+1}$ are carried into next period, more is at stake for the lender, as the potential losses in the event of default are larger. Hence, the probability mass on the default states would be even higher than before. One the other hand, higher $B_{t+1}$ also means higher default risk in the future, which would also depress next-period bond prices and thereby the payoff of the bond over the repayment interval. Since the optimal probability distortions are assigned on the basis of the relative payoff in each state, they may be higher or lower than before. While probability distortions (over the default interval) turn smaller in the top panels of Figure 2 as debt increases, the opposite occurs in the bottom ones.

By comparing the two panels in the left-side column (or the right-side column) we can see how the probability distortion changes with the level of current endowment. Due to the persistence of the stochastic process for $(y_t)_t$, the lower the current endowment, $y_t$, the lower the conditional mean of next period’s endowment, $y_{t+1}$ of the approximating density. For the state vectors considered here, the agent gets relatively more indebted as current endowment $y_t$ rises. This follows from the fact that output costs of default are increasing in the endowment $y_t$. The higher $y_t$, the more severely the borrower is punished if it defaults. As the incentives to default are smaller, the returns are lower, or equivalently the bond prices demanded by the lenders are higher, for the same levels of debt. Facing relatively cheaper debt, the borrower responds by borrowing more. In this way, more bond holdings $B_{t+1}$ widens the intervals of $y_{t+1}$-realizations for which default is optimal. At the same time, probability distortions over the new default interval become relatively larger for similar reasons as discussed previously when debt $B_t$ rises. In these cases, the perceived probability distortions, however, decrease due to the rightward shift of the conditional mean of $y_{t+1}$, as endowment $y_t$ increases.

Note also that for any value of $\theta \in (\theta, +\infty]$—with and without misspecification concerns— the (one-period) gross risk-free rate is nonstochastic and equal to the reciprocal of the lenders’ discount factor, i.e. $1 + r^f = \frac{1}{\gamma}$.

**One-period bonds.** Let us now analyze the case with one-period bonds, that is, $\lambda = 1$. For one-period debt, the pricing equation (7) for any $(y_t, B_{t+1})$ collapses to

$$q(y_t, B_{t+1}) = \gamma E_Y \left[ E_X [\delta^*(W_{t+1}, B_{t+1})] m^*(Y_{t+1}; y_t, B_{t+1}) | y_t] \right].$$

(8)

In this environment, the equilibrium bond prices turn to be the discounted probability, computed under distorted $\tilde{f}$, of not defaulting next period. To gain some insight on the asset-pricing implications of model uncertainty in this economy, we assume the output shock $x_t$ equals zero and
recast (8), for any \((y_t, B_t)\) and the equilibrium debt level \(B^*(y_t, B_t)\), as the standard asset-pricing equation:

\[
q(y_t, B^*(y_t, B_t)) = E_Y [SDF_{t+1}^* \times \delta_{t+1}^* | y_t] = \frac{1}{1 + r_f} (1 - P_Y \gamma \delta_{t+1}^* = 1 | y_t) + COV_Y [SDF_{t+1}^*, \delta_{t+1}^* | y_t],
\]

where \(SDF_{t+1}^* \equiv \gamma m_R^* (Y_{t+1}; y_t, B^*(y_t, B_t))\) stands for the market stochastic discount factor at \(t+1\), and the default indicator \(\delta_{t+1}^* \equiv \delta^* (Y_{t+1}, B^*(y_t, B_t))\) represents the payoff of the risky bond.

From the equation above it is necessary to have the covariance term sufficiently negative to generate low bond prices in equilibrium, or equivalently, high bond returns. This means that it is necessary that the stochastic discount factor typically be high when the borrower defaults. Lenders have to value consumption more when default events occur.

When \(\theta = +\infty\), the fears about model misspecification vanish and we are back to the case with risk-neutral lenders with rational expectations, as in Arellano (2008). Note that the stochastic discount factor is equal to \(\gamma\), and, therefore, the covariance between the market stochastic discount factor and the borrower’s default decisions is zero. This explains why it is not possible to generate high enough spreads in that particular environment.

With uncertainty-averse lenders, the covariance term is not zero anymore. The modified stochastic discount factor is then given by two multiplicative components, \(\gamma m_R^* (\cdot; y_t, B^*(y_t, B_t))\). As mentioned earlier, the probability distortion \(m_R^*\) is typically low when the borrower repays and high when the borrower defaults. It therefore induces a desired negative co-movement between the stochastic discount factor and the default decisions of the borrower.

A natural question is whether risk aversion on the lenders’ side with time separable preferences could generate a stochastic discount factor, negatively correlated with default decisions \(\delta_{t+1}^*\), that could help account for low bond prices, while preserving the default frequency at historical low levels. We explore this in Section 5.2 and Appendix B. Our findings indicate that in our calibrated economy with CRRA separable preferences for the lender this is not the case, that is, plausible degrees of risk aversion on the lenders’ side are not sufficient to generate high bond returns. See Table 8 for details. These results are also consistent with the findings by Lizarazo (2010) and Borri and Verdelhan (2010).\(^{30}\)

Even if sufficiently high values of risk aversion could eventually recover the high spreads shown in the data, doing so, however, would lower the risk-free rate to levels far below those exhibited in the data, in line with Weil (1989) risk-free rate puzzle.\(^{31}\)

In our environment with model uncertainty, however, the extent to which lenders are uncertainty-averse does not affect the equilibrium gross risk-free rate, given by the reciprocal of \(\gamma\), as their period

\(^{30}\)This is analogous to the equity premium puzzle result studied in Mehra and Prescott (1985).

\(^{31}\)Note that the stochastic process assumed for \(Y_t\) is stationary. If we add a positive trend, the risk-free rate would be rising, rather than decreasing, as the lenders’ coefficient of risk aversion goes up.
utility function is linear in consumption.\[^{32}\]

\[^{32}\]In our model with linear lenders’ per-period utility, equilibrium prices depend exclusively on economic fundamentals of the borrowing economy and the lenders’ preference for robustness. It is noteworthy to remark that adding curvature on the per-period utility will, in general, lead to equilibrium prices that also depend on international lenders’ characteristics such as their total wealth and investment flows, or more generally, on global macroeconomic factors, in line with the empirical findings by Longstaff et al. (2011). This seems to be an interesting extension to pursue in future research.

Figure 2: Approximating and distorted densities for different state vectors \((y_t, B_t)\).

4 Dynamics of Investors’ Wealth and Implications for Prices

In this section we show that we can relax some assumptions on the lenders’ problem without affecting key equilibrium objects. Besides the theoretical contribution, these results have useful implications for solving and calibrating the model.

In particular, we extend our setup in two important dimensions. We first consider a stochastic endowment for the lender and second we allow lenders to trade other financial assets beyond sovereign debt markets; the processes for the endowment or asset payoffs can be fully trusted or not by the lenders. We show that extending the framework in either direction, as long as the
lenders’ endowment and the payoff of the other financial assets are independent of the borrower’s endowment, would not alter the equilibrium prices nor the borrower’s allocations.

4.1 Stochastic Endowment for Investors

We first explore how equilibrium prices change when we assume a more general stochastic process for the endowment of the lender \( z_t \in \mathbb{Z} \), namely

\[
Z_{t+1} = \rho_0 + \rho_1 Z_t + \epsilon_{t+1},
\]

where \( \epsilon_{t+1} \) is distributed according to the cdf \( F_\epsilon(\cdot|y_{t+1}, y_t) \). Note that under this specification the endowments of the borrower and of the lenders can be correlated. For the ease of notation, we assume that both \( Y_t \) and \( Z_t \) are continuous random variables with conditional pdfs \( f_Y \) and \( f_Z \), respectively. Also, for simplicity we omit \( X_t \); the generalization that allows for it is straightforward.

We also allow the lenders to distrust the specification of the stochastic process of \( Z_t \), as well as that of \( Y \), but possibly to a different extent. Let \( \theta \) and \( \eta \) be the penalty parameters controlling for the degrees of concern about model misspecification for the distributions of \( y \) and \( z \), respectively. Different degrees of concern for each process are consistent with our view that there are more extensive, reliable datasets, especially from official statistical sources, containing relevant macro-financial information for developed economies and global capital markets, than for emerging economies.

Note that by exponentially twisting the probability of \( Z_t \) we induce some curvature in the lenders’ value function with respect to \( Z_t \), which would be embedded in the market stochastic discount factor for pricing any financial asset.

In this economy, to distort the expectation of the lenders’ continuation values, the minimizing agent will be placing two types of probability distortions, albeit not simultaneously. Indeed, first, it distorts the distribution of \( Z_{t+1} \) for each realization of \( Y_{t+1} \). Then, taking the resulting distorted continuation values for the lender as given, the minimizing agent proceeds to twist the probability of \( Y_{t+1} \).

The next theorem characterizes the equilibrium for this economy. For convenience, we define the following risk-sensitive operators: \( R_\theta \) and \( T_\eta \), where for any \( g \in L^\infty(Y) \),

\[
R_\theta[g](y) = -\theta \log E_Y \left[ \exp \left\{ -\frac{g(Y')}{\theta} \right\} \right] |y|
\]

for any \( y \), and for any \( h \in L^\infty(Z) \),

\[
T_\eta[h](y', y, z) = -\eta \log E_Z \left[ \exp \left\{ -\frac{h(y', Z')}{\eta} \right\} \right] |y', y, z|
\]

for any \( (y', y) \), and where \( E_Z[\cdot|y', y, z] \) is the conditional expectation of \( Z_{t+1} \), given \( (y_{t+1}, y_t, z_t) = \)
(y', y, z).

These risk-sensitive operators summarize the minimization problem for the probability distortions; the continuation value for the lender is a nested combination of them.

**Theorem 4.1.** There exists a recursive equilibrium for this economy such that the equilibrium price function is given by:

\[
q^o(y_t, B_{t+1}) = \gamma E_Y [(\lambda + (1 - \lambda)(\psi + q^o(Y_{t+1}, B^o(W_{t+1}, B_{t+1}))))\delta^o(Y_{t+1}, B_{t+1})m^o(Y_{t+1}; y_t, B_{t+1})] 
\]

for any \((y_t, B_{t+1})\), where: (i) For any \(y_{t+1}\),

\[
m^o(y_{t+1}; y_t, B_{t+1}) \equiv \exp \left\{ \frac{T_{\gamma(1-\gamma\rho_1)}[\epsilon_{t+1}|yt+1]}{1-\gamma\rho_1} + \tilde{W}^o(y_{t+1}, B_{t+1}, B_{t+1}) \right\} E_Y \left[ \exp \left\{ -\frac{T_{\gamma(1-\gamma\rho_1)}[\epsilon_{t+1}|yt+1]}{1-\gamma\rho_1} + \tilde{W}^o(y_{t+1}, B_{t+1}, B_{t+1}) \right\} | y_t \right] 
\]

(ii) \((B^o, \delta^o)\) correspond to the optimal policy functions in the borrower’s problem, given \(q^o\); and (iii)

\[
\tilde{W}^o(y, B, B) \equiv \delta^o(y, B)\tilde{W}^o_R(y, B, B) + (1 - \delta^o(y, B))\tilde{W}^o_A(y),
\]

where \((\tilde{W}^o_R, \tilde{W}^o_A)\) solve the following problem

\[
\tilde{W}^o_R(y_t, B_{t+1}, b_t) = \max_{b_{t+1}} \left\{ \{q^o(y_t, B_{t+1})(b_{t+1} - (1 - \lambda)b_t - (\lambda + (1 - \lambda)\psi)b_t) \right. \\
+ \gamma R \left[ \frac{T_{\gamma[1-\gamma\rho_1]}[\epsilon_{t+1}|yt+1]}{1-\gamma\rho_1} + \tilde{W}(Y_{t+1}, B_{t+1}, b_{t+1}) \right] (y_t) \}, 
\]

and

\[
\tilde{W}^o_A(y_t) = \gamma R \left[ \frac{T_{\gamma[1-\gamma\rho_1]}[\epsilon_{t+1}|yt+1]}{1-\gamma\rho_1} + (1 - \pi)\tilde{W}_R(Y_{t+1}) + \pi\tilde{W}(Y_{t+1}, 0, 0) \right] (y_t),
\]

We relegate the proof to Appendix E. A few remarks about the theorem are in order. First, the borrower’s optimal policy functions \((\tilde{W}^o_R, \tilde{W}^o_A)\) do not depend on \(z_t\). This is because the price function does not depend on \(z_t\) and thus the borrower does not need to keep track of it in order to predict future prices.

Second, by inspection of equation (12) we can formulate the following corollary

**Corollary 4.1.** If \(\epsilon_{t+1}\) is independent of \((Y_t)_t\), i.e., \(F_t(\cdot|yt+1, y_t) = F_t(\cdot)\), then \(q^o = q\) and \((B^o, \delta^o) = (B, \delta)\).
That is, if $\epsilon_{t+1}$ is independent of $(Y_t)_t$, then the equilibrium price function and debt and default decisions are identical to those in our economy. This is due to the fact that, although randomness of $Z_t$ affects the level of the utility of the lender, it does not affect his marginal decisions. This result is independent of any misspecification doubts about the process of $Z_t$ and implies that our results remain unchanged once we allow for the more general stochastic process for $(Z_t)_t$ provided it is independent of $(Y_t)_t$.\(^{33}\) It also follows that the relative size of the endowment for lenders is irrelevant for equilibrium bond prices and the borrower’s allocations.

An important implication of these results for our calibrations is that there is no need to identify who the lenders are in the data, and, in particular, to find a good proxy of their income relatively to the borrower’s endowment.

4.2 Implications of Introducing Additional Risky Assets in our Economy

In what follows we introduce a new asset in the economy to be traded by the lenders. Let $(D_{t+1})_t$ be an (first order) Markov stochastic process, independent of $(Y_t)_t$; $d_{t+1} \geq 0$ is the payoff realization at time $t + 1$, of the new asset. For simplicity we assume that its aggregate supply $\Xi_t$ is deterministic; results can easily be extended to more general specifications of $\Xi_t$. Also, let $p(d_t)$ be the price of one unit of this asset at time $t$ when the realization of the payoff is $d_t$. Finally, we use $\xi_t$ to denote the lender’s individual position on the asset at time $t$.

We assume that the lender has concerns about misspecification in the law of motion for $D_t$. Abusing notation, we use $\eta$ to denote the parameter controlling the degree of robustness over the process for $D_t$.

We call the economy with this additional asset, the extended economy. The goal of this section is to characterize the recursive equilibrium of the extended economy. We use superscript “e” to denote the variables that form a recursive equilibrium of the extended economy, for example, $q^e$ is the equilibrium price function of the extended economy, and so on. Observe that in the extended economy, the aggregate state is given by $(y_t,B_t,d_t)$ and the state for the lender’s problem $(y_t,B_t,b_t,d_t,\xi_t)$.

\(^{33}\)As reported in Shapiro and Pham (2006), Italy, Switzerland and the U.S. were the three countries with largest foreign holdings of Argentinean debt defaulted in 2001, accounting for 25, 17 and 15 percent, respectively, of the total debt held by foreign investors.

We estimate a Gaussian VAR(1) for $Y$ and $Z$ using detrended log real GDP of Argentina and a detrended real GDP for each of the three creditor economies, respectively; our sample spans from 1993:Q1 to 2001:Q3. We find that the p-values of a test for jointly non-significance of the coefficients of $Z_{t-1}$ for $Y_t$ and $Y_{t-1}$ for $Z_t$, and for the covariance of residuals, are equal to 31.6 (Italy), 29.7 (Switzerland) and 3.0 percent (U.S.). These result imply that, for our sample and econometric specification, we cannot reject the assumption of independence of $Y$ and $Z$ at a 5-percent confidence level, except for the U.S.
Theorem 4.2. There exists a recursive equilibrium of the extended economy such that

\[ W^R_R(y_t, B_t, b_t, d_t, \xi_t) = W_R(y_t, B_t, b_t) + \mathcal{U}(d_t, \xi_t) \]

and \( W^A_A(y_t, d_t, \xi_t) = W_A(y_t) + \mathcal{U}(d_t, \xi_t) \),

where \( \mathcal{U} \) solves

\[ \mathcal{U}(d_t, \xi_t) = \max_{\xi_{t+1}} \{ p^e(d_t) \xi_{t+1} - d_t \xi_t - \gamma \eta \log E_D \left[ \exp \{-\eta^{-1}\mathcal{U}(D_{t+1}, \xi_{t+1})\}|d_t]\} \cdot \]

And \( q^e = q \), \( B^e = B \), \( \delta^e = \delta \) and

\[ p^e(d_t) = \gamma E_D [D_{t+1} n(D_{t+1}; d_t, \Xi_{t+1}) | d_t], \quad (17) \]

with

\[ n(d_{t+1}; d_t, \Xi_{t+1}) \equiv \frac{\exp\{-\eta^{-1}\mathcal{U}(d_{t+1}, \Xi_{t+1})\}}{E_D[-\eta^{-1}\mathcal{U}(D_{t+1}, \Xi_{t+1})|d_t]}. \]

The proof is relegated to the Appendix E. The intuition behind it, is that due to the linear per-period payoff and independence of \( D_t \) and \( Y_t \) we can separate the portfolio problem of choosing \( b_{t+1} \) and \( \xi_{t+1} \) into two independent problems. The former problem is identical to the one in our benchmark economy.

According to the theorem, if we introduce a new asset whose payoff is independent of the borrower’s endowment, the equilibrium price for the sovereign bond and the borrower’s allocations remain unchanged.\(^{34}\) A particular asset of interest is a risk-free one, i.e., \( D = 1 \), and \( p^e(d_t) = \gamma \), which would capture the case in which lenders are allowed to invest, say, in U.S. Treasury bills. An important implication of this result is that, when solving the model numerically, we do not need to keep track of the total wealth of lenders beyond that consisting of risky bonds.

A few remarks about these results are in order. First, the stochastic discount factor in the extended economy is now given by \( \gamma m^*_R(y_{t+1}; y_t, B_t)n^*(d_{t+1}; d_t, \Xi_{t+1}) \), where \( n^* \) is the probability distortion over the distribution of next-period \( D_{t+1} \). Because the asset payoff does not depend on \( Y_{t+1} \), we are able to integrate out \( m^*_R \) in equation (17). Second, note that equation (17) is a standard risk-neutral pricing under model uncertainty. An equivalent version but with a pricing kernel enlarged by the ratio of consumption levels (due to logarithmic period utility) can be found for example in Barillas et al. (2009).

Finally, albeit outside the scope of the current paper, we view extending this result to the case

\(^{34}\)We conducted a similar VAR(1) estimation to that in the previous subsection, but using detrended S&P real payoff instead of detrended real GDP for creditor economies. In this case, the p-value of the same test of jointly non-significance is 4.8 percent.
where $D$ and $Y$ are not independent an interesting avenue for future research. In this case the demand for sovereign debt (and $\theta$, in particular) will influence the price $p^e$, and this will impose further restrictions on the value for $\theta$ that go above and beyond the ones imposed in this paper using detection error probabilities in Section 7.

5 Quantitative Analysis

In this section we analyze the quantitative implications of our model for Argentina. To do so, we specify our choices for functional forms and calibrate some parameter values to match key moments in the data for the Argentinean economy. The period considered spans from the first quarter of 1993, when Argentina regained access to financial market with the Brady Plan agreement after its 1982 default, to the last quarter of 2001, when Argentina defaulted again on its foreign debt.

5.1 Calibration

In Table 1 we present the parameter values for the calibration of our benchmark model.

For the quantitative analysis, we consider the following functional forms. The period utility function for the borrower is assumed to have the CRRA form, i.e., $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ where $\sigma$ is the coefficient of relative risk aversion.

We assume that the endowment of the borrower follows a log-normal AR(1) process,  
$$\log Y_{t+1} = \rho \log Y_t + \sigma \epsilon_{t+1},$$
where the shock $\epsilon_{t+1} \sim \mathcal{N}(0, 1)$. As shown in Theorem 4.1, the (non-stochastic) lenders’ endowment does not affect the equilibrium bond prices and borrower’s allocations, allowing us to circumvent the subtle challenge of providing a good proxy for lenders’ consumption or income. We therefore set the lenders’ logged endowment, denoted by $\log(z)$, to 1.

Following Chatterjee and Eyingungor (2012), we consider the specification for output costs

$$\phi(y) = \max \{0, \kappa_1 y + \kappa_2 y^2\},$$

with $\kappa_2 > 0$. As explained later, our calibrated output costs play a key role in generating desired business cycle features for emerging economies, in particular the volatility of bond spreads, in the context of lenders’ model uncertainty.

The coefficient of relative risk aversion for the borrower $\sigma$ is set to 2, which is standard in the sovereign default literature. The re-entry probability $\pi$ is set to 0.0385, implying an average period
of 6.5 years of financial exclusion and consistent with Benjamin and Wright (2009) estimates.\footnote{Pitchford and Wright (2012) report an average 6.5-year delay in debt restructuring after 1976.}

We estimate the parameters $\rho$ and $\sigma_r$ for the log-normal AR(1) process for the endowment of the borrower, using output data for Argentina for the period from 1993:Q1 to 2001:Q3.\footnote{We exclude the last quarter of 2001 since the default announcement by President Rodriguez Saá took place on December 23, 2001.} Time series at a quarterly frequency for output, consumption, and net exports for Argentina are taken from the Ministry of Finance (MECON). All these series are seasonally adjusted, in logs, and filtered using a linear trend. Net exports are computed as a percentage of output.

The (one-period) risk-free rate $r^f$ in the model is 1 percent, which is approximately the average quarterly interest rate of a three-month U.S. Treasury bill for the period in consideration. As stated before, the lenders’ discount factor $\gamma$ is set equal to the reciprocal of the gross risk-free rate $1 + r^f$. The parameters governing the payoff structure of the long-term bond, $\lambda$ and $\psi$, are chosen to replicate a median debt maturity of 5 years and a coupon rate of 12 percent.

Bond spreads are computed as the difference between annualized bond returns and the U.S. Treasury bill rate. The quarterly time series on interest rate for sovereign debt for Argentina is taken from Neumeyer and Perri (2005). To calculate the yield of the long-term bond, we use the internal rate of return.\footnote{The internal rate of return of a bond, denoted by $r(y_t, B_{t+1})$, is determined by the pricing equation: $q(y_t, B_{t+1}) = \frac{\lambda + (1 - \lambda)\psi}{\lambda + r(y_t, B_{t+1})}$.}

We calibrate the parameters $\beta$, $\kappa_1$, $\kappa_2$, and $\theta$ in our model to match key moments for the Argentinean economy. We set the borrower’s discount factor $\beta$ to target an annual frequency of default of 3 percent. The calibrated value for $\beta$ is 0.9627, which is relatively large within the sovereign default literature.\footnote{For example, Yue (2010) uses a discount factor of 0.74 and Aguiar and Gopinath (2006) use 0.80.}

We select the output cost parameters $\kappa_1$ and $\kappa_2$ to match the average debt level of 46 percent of GDP for Argentina and the spreads volatility of 4.58 percent.\footnote{The foreign government debt to output ratio of 46 percent for Argentina is taken from the National Institute of Statistics and Censuses (INDEC) for the period from 1994:Q4 to 2001:Q4.}

Regarding the degree of model uncertainty in our economy, we take the following strategy: we first set the penalty parameter $\theta$ to match the average bond spreads of 8.15 percent observed in the data for Argentina. As explained in Section 7, the value of $\theta$ is itself not necessarily informative of the amount of distortion in lenders’ perceptions about the evolution of $y$. The impact of the value of $\theta$ on probability distortions is context-specific.\footnote{See Barillas et al. (2009) for a simple example with a random walk model and a trend stationary model for log consumption.} To better interpret our results, we provide another statistic, the detection error probabilities (DEP), commonly used in the robust control literature.\footnote{See Anderson et al. (2003), Maenhout (2004), Barillas et al. (2009), Bidder and Smith (2011), and Luo.
In our economy lenders are concerned about alternative models that they find hard to differentiate statistically from one another, given the available series of output data. To measure how close two competing models are, and therefore, how difficult is to distinguish between them, we use DEP. DEP gauges the likelihood of selecting the incorrect model when discriminating between the approximating and the worst-case model using likelihood ratio tests; for details on its computation see Section 7. The lower the value of DEP, the more pronounced is the discrepancy between these two models. If they are basically identical, they are indistinguishable and hence the DEP is 0.50. In contrast, if the two models are perfectly distinguishable from each other, the DEP is 0. Barillas et al. (2009) suggest 20 percent as a reasonable threshold, in line with a 20-percent Type I error in statistics. In our model the DEP implied by our calibrated $\theta$ is only 31 percent, which implies that around one third of the time the detection test indicates the wrong model. This value is therefore quite conservative, suggesting that only a modest amount of model uncertainty is sufficient to explain the high average bond spreads observed in the data.

**Computational algorithm.** The model is solved numerically using value function iteration. To that end, we apply the discrete state space (DSS) technique. The endowment space for $y_t$ is discretized into 200 points and the stochastic process is approximated to a Markov chain, using et al. (2012), for example.
Tauchen and Hussey (1993) quadrature-based method.\footnote{For bond holdings, we use 580 gridpoints to solve the model and no interpolation. Also, the distribution for $x_t$ is truncated between $[-2\sigma_x, 2\sigma_x]$.}

When solving the model using the DSS technique, we may encounter lack of convergence problems; see Chatterjee and Eyingungor (2012) for details. To avoid that for long-term debt, we introduce the i.i.d. continuous output shock $X_t$. For the case of short-term debt, we follow a different approach described in Subsection 5.3.

To compute the business cycle statistics, we run 2,000 Monte Carlo (MC) simulations of the model with 4,000 periods each.\footnote{To avoid dependence on initial conditions, we pick only the last 2,000 periods from each simulation. The unconditional default frequency is computed as the sample mean of the number of default events in the simulations.} Similarly to Arellano (2008), to replicate the period for Argentina from 1993:Q1 to 2001:Q3, we consider 1,000 sub-samples of 35 periods with access to financial markets, followed by a default event.\footnote{Because Argentina exited financial autarky with the Brady bonds while in our model it does so with no debt obligations, we also impose no reentry in the previous four quarters (1 year) of each candidate sub-sample.} We then compute the mean statistics and the 90-percent confidence intervals, across MC simulations, for these subsamples.

\textbf{Output costs and implications.} The choice of Chatterjee and Eyingungor (2012) specification for the output costs of default given by expression (18) is key for matching some business cycle moments.

Similar to their calibration, we have $\kappa_1 < 0$, which implies that there are no output costs for realizations $y < \kappa_2/\kappa_1$, and the output costs as a fraction of output increase with $y$ for $y > \kappa_2/\kappa_1$. In this sense, our output costs are similar to those in Arellano (2008), both of which have significant implications for the dynamics of debt and default events in the model. As explained before, when output is high, there is typically less default risk, bond returns are low and there is more borrowing. For low levels of output, the costs of default are lower, hence, the default risk is higher, and so are the bond returns. If the borrower is hit by a sequence sufficiently long of bad output realizations, it eventually finds it optimal to declare default.

As noted by Chatterjee and Eyingungor (2012), this functional form for output costs has an important advantage over those of Arellano (2008) for the volatility of bond spreads. In Arellano (2008) output costs as a fraction of output vary significantly with output, and so do the default incentives.\footnote{Quantitatively, in that specific framework with one-period bonds, spreads volatility can be considerably reduced when using very fine grids or alternative computational methods to solve the model numerically, as shown by Hatchondo et al. (2010). For long-term debt models, however, no comparison between solution methods has been driven.} Hence, the default probability is very sensitive to the endowment realizations $y$. In our model, the sensitivity of the distorted default probability is even higher. Beliefs’ distortions play out in the same direction, by slanting probabilities even more towards the range of endowment.
realizations in which default occurs. As a result, not surprisingly, the variability of bond spreads rises significantly when we introduce doubts about model misspecification.

For this reason, instead of using the output cost structure from Arellano (2008), we consider the specification given by (18). In this case, the output loss as a proportion of output is less responsive to fluctuations of $y$. It therefore yields a lower sensitivity of default probabilities to $y$, reducing at the same time the spreads volatility.

5.2 Simulation Results

Table 2 reports the moments of our benchmark model and in the data.\textsuperscript{46} For comparison purposes, it also shows the corresponding moments for Chatterjee and Eyingungor (2012), denoted CE model, probably the best-performing long-term debt model in the literature, and the “baseline model”, which is a re-calibrated version of our model but without model uncertainty. The latter is contemplated because we are particularly interested in checking how the model performs when we anchor the default frequency to 3 percent per year.\textsuperscript{47}

Overall, the model matches standard business cycle regularities of the Argentinean economy. More importantly, we can replicate salient features of the bond spreads dynamics. By introducing doubts about model misspecification, we can account for all the average bond spreads observed in the data, as well as their volatility, matching at the same time the historical annual frequency of default of 3 percent and the average risk-free rate. An important contribution of our paper is that we only require quite limited amount of model uncertainty to do it. Indeed, we need on average smaller deviations of lenders’ beliefs to explain the spread dynamics than those used in the equity premium literature.\textsuperscript{48,49}

Notably, our model can explain the average bond spreads of 8.15 percent in the data, which is roughly three percentage points higher than the 5.01 percent obtained by the baseline model. In our environment, risk-neutral lenders charge an additional uncertainty premium on bond holdings

\textsuperscript{46}While consumption is relatively more volatile than output in the data over the entire period 1983-2001, as reported by Arellano (2008), it is not the case if we restrict the sample to 1993-2001.

\textsuperscript{47}While the theoretical model in Chatterjee and Eyingungor (2012) and our ”baseline model” are identical, their calibrations differ along several dimensions. Besides targeting different moments in the data, a different parametrization is considered for the endowment process, as well as different number of asset gridpoints and sampling criterion are used. In the calibration of the ”baseline model”, in contrast with Chatterjee and Eyingungor (2012), the discount factor is set to match an annual default frequency of 3 percent, while the output cost parameters, $\kappa_1$ and $\kappa_2$, are chosen to deliver a debt-to-output ratio of 46 percent, and a spreads volatility as close as possible to the level observed in the data.

\textsuperscript{48}To explain different asset-pricing puzzles, Maenhout (2004), Drechsler (2012), and Bidder and Smith (2011) require detection error probability in the range between 10 and 12 percent. Barillas et al. (2009) need even lower values to reach the Hansen and Jagannathan (1991) bounds.

\textsuperscript{49}It is worth noting that while we assume no recovery on defaulted debt in the model—which in equilibrium pushes up the bond returns—, there is room to increase the amount of model uncertainty (i.e., decrease $\theta$) within the plausible range, and thus we could still account for the bond spread average level if any mechanism of debt restructuring with subsequent haircuts is introduced.
Table 2: Business Cycle Statistics for our Model, Chatterjee and Eyingungor (2012), the Data and the Baseline model. The 90-percent confidence interval generated by the MC simulations are reported in parenthesis.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Data</th>
<th>CE Model</th>
<th>Baseline Model</th>
<th>Our Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean((r - r_f))</td>
<td>8.15</td>
<td>8.15</td>
<td>5.01 (3.0,7.3)</td>
<td>8.15 (5.7,11.1)</td>
</tr>
<tr>
<td>Std.dev.((r - r_f))</td>
<td>4.58</td>
<td>4.43</td>
<td>4.27 (1.8,7.4)</td>
<td>4.62 (1.9,7.8)</td>
</tr>
<tr>
<td>mean((-b/y))</td>
<td>46</td>
<td>70</td>
<td>42 (34,48)</td>
<td>44</td>
</tr>
<tr>
<td>Std.dev.((c)/std.dev. (y))</td>
<td>0.87</td>
<td>1.11</td>
<td>1.16</td>
<td>1.23</td>
</tr>
<tr>
<td>Std.dev.((tb/y))</td>
<td>1.21</td>
<td>1.46</td>
<td>0.89 (0.6,1.2)</td>
<td>1.23 (0.9,1.6)</td>
</tr>
<tr>
<td>Corr((y,c))</td>
<td>0.97</td>
<td>0.99</td>
<td>0.99 (0.98,1.0)</td>
<td>0.98 (0.95,1.0)</td>
</tr>
<tr>
<td>Corr((y, r - r_f))</td>
<td>-0.72</td>
<td>-0.65</td>
<td>-0.78 (-0.9,0.6)</td>
<td>-0.75 (-0.9,-0.5)</td>
</tr>
<tr>
<td>Corr((y, tb/y))</td>
<td>-0.77</td>
<td>-0.44</td>
<td>-0.80 (-1.0,-0.5)</td>
<td>-0.68 (-0.9,-0.3)</td>
</tr>
<tr>
<td>Drop in y (around default)</td>
<td>-6.4</td>
<td>-4.5</td>
<td>-3.9</td>
<td>-5.6</td>
</tr>
<tr>
<td>DEP</td>
<td>NA</td>
<td>NA</td>
<td>50.0</td>
<td>31.3</td>
</tr>
<tr>
<td>Default frequency (annually)</td>
<td>3.00</td>
<td>6.60</td>
<td>3.00 (1.6,4.6)</td>
<td>3.00 (1.6,4.6)</td>
</tr>
</tbody>
</table>

To get compensated for bearing the default risk under the worst-case density for output. In turn, their *perceived* conditional probability of default next period—while having access to financial markets—is on average 2.2 percent per quarter, while the actual one is only 0.9 percent. Lenders’ distorted beliefs about the evolution of the borrowing economy enable us to achieve the challenging goal of simultaneously matching the low sovereign default frequency and the high average level (and volatility) of excess returns on Argentinean bonds exhibited in the data. Additionally, our model can account for a strong countercyclicality of bond spreads.

As shown in Table 2, Chatterjee and Eyingungor (2012) (CE model) has also been able to match the average bond spreads observed in the data. To our knowledge, only this paper and Hatchondo et al. (2010) have been able to do that under rational expectations. These authors, however, reproduce the average high spreads for Argentina at the cost of roughly doubling the default frequency to around 6 percent annually. While it is difficult to determine what the true value for the default frequency is in the data, it seems to be consensus in the literature that it lies close to 3 percent per year (see footnote 7). These papers and ours replicate this feature of the bond spreads in a general equilibrium framework. In contrast, Arellano (2008) and Arellano and Ramanarayanan (2012)\(^{50}\), have been able to account for the bond spreads dynamics by assuming an ad hoc functional form for the stochastic discount factor, which depends on the output shock to the borrowing economy. Our paper can be seen as providing microfoundations for such a functional form. Section 6 elaborates on this point.

In order to shed more light on the behavior of the spreads, we report in Table 3 different\(^{50}\) See also and Hatchondo et al. (2010).
percentiles. In all cases, our model’s 90-percent confidence intervals contain the value observed in the data, and the average across MC simulations is very close to the one observed in the data. The baseline model, however, yields percentiles that are considerably below the values observed in the data. Finally, we note that the median is always lower than the average in the data and in the models, due to the presence of large peaks because of the default events. These results show how our model is able to match the average level, volatility and countercyclicality of spreads, while not distorting other relevant moments.

Also, the introduction of plausible degrees of risk aversion on the lenders’ side with time-separable preferences has shown to be insufficient to recover the high spreads observed in the data. With constant relative risk aversion, as in Lizarazo (2010), matching high spreads calls for a very large risk aversion coefficient, in line with Mehra and Prescott (1985), and implausible risk-free rates, as pointed out by Weil (1989) in the context of studies on the equity premium. In appendix B we show simulations that quantify these facts for this setting.\footnote{Borri and Verdelhan (2010) have studied the setup with positive co-movement between lenders’ consumption and output in the emerging economy in addition to time-varying risk aversion on the lenders’ side. To generate endogenous time-varying risk aversion for lenders, they endow them with Campbell and Cochrane (1999) preferences with external habit formation. However, they find that even with these additional components average bond spreads generated by the model are far below from those in the data. They report average bond spreads of 4.27 percent, for an annual default frequency of 3.11 percent.}

Our model can also generate considerable levels of borrowing, consistent with levels observed in the data. High output costs of default jointly with a low probability of regaining access to financial markets imply a severe punishment to the borrower in case it defaults. Consequently, higher levels of indebtedness can be sustained in our economy. Since the magnitude of output cost can be hard to gauge from the parameter values $\kappa_1$ and $\kappa_2$, we report the average output drop that the borrowing economy suffers in the periods of default announcements. We then compare this statistic with the actual contraction in Argentinean output observed in the data around the fourth quarter of 2001.\footnote{To be consistent with our model, the same linear trend from the estimation was employed when computing the actual drop of output in the data.}\footnote{The drop observed in 2002Q1 was 7.3 percent, slightly larger than in the previous quarter.}

\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
Statistic & Data & Baseline Model & Our Model \\
\hline
$Q_{0.10}(r - r^f)$ & 4.40 & 2.12 & (1.2,3.2) & 4.66 & (3.3,6.1) \\
$Q_{0.25}(r - r^f)$ & 5.98 & 2.62 & (1.4,3.9) & 5.38 & (3.9,7.0) \\
$Q_{0.50}(r - r^f)$ & 7.42 & 3.57 & (1.9,5.5) & 6.66 & (4.7,8.9) \\
$Q_{0.75}(r - r^f)$ & 8.45 & 5.55 & (2.9,9.4) & 9.08 & (5.9,13.5) \\
$Q_{0.90}(r - r^f)$ & 11.64 & 9.65 & (4.4,18.6) & 13.61 & (7.6,23.1) \\
\hline
\end{tabular}
\end{center}
\caption{Quantiles of spreads for our Model, the Data and the Baseline Model. $Q_\alpha(r - r^f)$ denotes the $\alpha$-th quantile. The 90-percent confidence interval generated by the MC simulations are reported in parenthesis.}
\end{table}
is quite in line with the data.

Finally, our model reproduces quantitatively standard empirical regularities of emerging economies: strong correlation between consumption and output, and volatility and countercyclicality of net exports. Along these dimensions, our model performs similarly to that of Chatterjee and Eyingungor (2012) and the baseline model.

5.3 Long-term Bonds vs. One-period Bonds

Introducing long-term bonds has significant implications on equilibrium bond prices and allocations. To highlight the contribution of our debt maturity choice in our model, we analyze in this section an economy with one-period bonds, i.e. \( \lambda = 1 \), and compare it with our long-term debt model.

5.3.1 Quantitative Findings

To study the quantitative implications of one-period debt in our economy, we set \( \lambda \) equal to one and recalibrate our model. To do so, we assume the same functional forms and use the same sampling criterion as in Subsection 5.1.

In order to compare with the standard models with one-period debt (e.g. Arellano (2008)), we set the output shock \( X_t \) equal to zero. However, lack of convergence problems still emerge when solving these models using discrete state space (DSS) technique. To handle them, we propose an approach based on an i.i.d. preference shock described below. An advantage of the latter approach is that if the variance of the preference shock converges to zero, our economy gets close to that without it; see Appendix F for the proof. Therefore, in our numerical simulations, by gradually reducing the volatility of the preference shock, we can analyze a one-period bond economy similar to Arellano (2008).

The parameter values needed to match the targeted moments are reported in Table 9 in Appendix C. In contrast with long-term debt, an extremely low discount factor is required to generate enough default episodes. In this case, the value for \( \beta \) is 0.7249. In our view, this implausibly high degree of impatience of the borrowing economy seems hard to reconcile with the data. Whether other relevant non-economic aspects, not modeled explicitly in our economy, such as political uncertainty, could rationalize the low value for \( \beta \) is outside the scope of this paper.

Nevertheless, note that the value for the discount factor needed to match the default frequency is low as a consequence of the different asset structure and not because of the presence of doubts about model misspecification. Indeed, when considering one-period bonds, Chatterjee and Eyingungor (2012) require a similar value for the discount factor in their rational-expectations environment. As explained there, when the borrower issues short-term debt it would be less willing to extend borrowing into the region with sizable default risk. This basically follows from the fact that the
marginal increase in net revenues for an additional unit of bond issued would typically be smaller with one-period bonds.

A high value for $\kappa_1$ is necessary to deliver higher output costs in 'good times', reducing the incentives to default in those states and hence the cost of debt, which thereby stimulates more borrowing.

As shown in Table 5, the model with one-period debt can also account for the bond spreads dynamics. To do so, however, a significantly larger amount of model uncertainty, reflected in a lower DEP, is necessary. While the DEP hovers around 0.31 with long-term debt, it drops to 0.27 with one-period bonds, which is still deemed as an acceptable level. This should not be surprising given that without model misspecification the long-term debt model delivers higher average spreads than the one-period bond model, for the same default frequency, as reported in Tables 2 and 5.

Also, the model displays excess volatility of consumption and net exports. Replicating the indebtedness level jointly with the average bond returns in a one-period bond economy necessarily translates into large capital outflows for interest repayments, as a fraction of output. Since the bond price is very sensitive to fluctuations in output $y_t$, interest repayments turn to be also very volatile, and so are consumption and net exports.

5.3.2 Computational algorithm

As pointed out by Chatterjee and Eyingungor (2012), solving sovereign default models using the discrete state space (DSS) technique may encounter convergence problems in some particular environments. Some one-period debt models, like the one considered in Section 5.3, are not the exception.

In order to address this technical complication, we propose here an alternative approach to that considered by Chatterjee and Eyingungor (2012) based on an output shock. We think our methodology could be of independent interest and extended to other settings. Our approach is based on the introduction of an i.i.d. preference shock, denoted as $\nu \sim F_\nu$, in line with McFadden (1981) and Rust (1994). We add $\nu$ only to the autarky utility of the borrower when it decides whether to default on the debt; but we do not add any shock when the economy is already in financial autarky. The shock $\nu$ could be interpreted as an error of an agent who intends to behave according to a certain payoff, but she incorrectly calculates the payoff by adding a noise.

We call the economy with the shock, the *perturbed economy*; we relegate the formal details of this new economy to Appendix F.

While the computational approach proposed by Chatterjee and Eyingungor (2012) may be more effective in dealing with convergence issues in some sovereign default models, particularly

54 Alternatively, we could have added the preference shock to the repayment utility of the borrower when making the default decision; due to the nature of the shock distribution, the results would remain unchanged.
with long-term debt, the method provided here may have some advantages for other environments.

First, we show that as the distribution of the preference shock $\nu$ is more concentrated around zero, then the limit of the equilibrium in the perturbed economy (if it exists), is an equilibrium described in Section 2.7. In line with this theoretical result, by reducing the volatility of $\nu$, we might be able to eventually study equilibrium prices and allocations in a more traditional framework without preference (and i.i.d. output) shocks, as in Arellano (2008) and Aguiar and Gopinath (2006).

Second, by carefully choosing the distribution for $\nu$ we can obtain closed forms for continuation values with access to financial markets and for the default indicator. This has important implications in terms of computation, since it does not rely on constructing an additional grid for the i.i.d. shock, which could slow down the algorithm considerably, and could introduce additional numerical/approximation errors.

For our numerical results, we postulate $F_\nu$ to be a logistic($h$) distribution, i.e., $F_\nu(v) = \frac{1}{1+\exp\left(-h^{-1}v\right)}$, where the parameter $h$ controls for the variance of the distribution. Note that, as $h \to 0$, then $F_\nu$ converges to a mass point at zero. Under this particular choice, it follows that the Bellman equation of the borrower when it decides to repay is given by

$$V_R(y_t, B_t) = \max_{B_{t+1} \in \mathbb{B}} \left\{ u(c_t) + \beta E_Y \left[ (h \log \left( 1 + \exp\left\{ \frac{V_A(Y_{t+1}) - V_R(Y_{t+1}, B_{t+1})}{h} \right\} \right) + V_R(Y_{t+1}, B_{t+1}) \right] | y_t \right\}.$$ 

The value function $V_A$ changes in a similar fashion. The default policy function turns out to be a probability,

$$(y_t, B_t) \mapsto \frac{1}{1 + \exp\{h^{-1}(V_A(y_t) - V_R(y_t, B_t))\}}. \tag{19}$$

5.4 A Graph for the Argentinian Case

In order to showcase the dynamics generated by our one-period and long-term debt models, we perform the following exercise. We input into the models the output path observed in Argentina for 1993:Q1 to 2001:Q4. Given this and an initial level of debt, each of the models generates a time series for the annualized spread and for one-step-ahead conditional probabilities of default under both the approximating and distorted models. Figure 3 and 4 correspond to the long-term debt and one-period bond economies, respectively. The top panel shows the output path, jointly with the time series for bond spreads exhibited in the data and delivered by our model. For comparison, we also plot the spreads generated by each of the baseline models. The bottom panel displays the conditional default probabilities according to our model.

---

$^{55}$We prove these results by extending Proposition 4 in Doraszelski and Escobar (2010) to our setting.

$^{56}$See Assumption F.1 in the Appendix F for the precise definition of convergence.
Figure 3: Top Panel: Output for Argentina; spreads generated by our long-term debt model and the baseline model; and actual spreads (measured by the EMBI+). Bottom Panel: One-step-ahead conditional probabilities of default under the distorted model and the approximating model.
Figure 4: Top Panel: Output for Argentina; spreads generated by our one-period debt model and the baseline model; and actual spreads (measured by the EMBI+). Bottom Panel: One-step-ahead conditional probabilities of default under the distorted model and the approximating model.
In each case, we can see that our model does a better job matching the actual spreads than the corresponding baseline model. The difference between the spreads generated by the two models can be largely explained by the behavior over time of one-step-ahead conditional probabilities of default. While these default probabilities are always positive and sizable with one-period bonds, they are not so when the borrower issues only long-term debt. For the latter case, we observe zero or negligible default risk right before and after the year of 1995. A salient feature of long-term debt models is that they can generate considerable bond spreads even in the absence of default risk in the near future. Lenders typically demand high returns on their long-term debt holdings to get compensated for possible capital losses due to future defaults on the unmatured fraction of their bonds. Additionally, further compensation is required for potential drops in the future market value of outstanding bonds as the borrower might dilute its debt.

In any case, when non-negligible, the subjective probability of default next period is higher than the actual one. More importantly, the wedge between the two probabilities is greater when output is low (and default is more likely next period), both for one-period and long-term debt, e.g. see results for 1995:Q2 to 1995:Q4 and from 2000:Q2 onwards.

Finally, it is worth pointing out that our results are consistent with the findings by Zhang (2008). Using CDS price data on Argentinean sovereign debt at daily frequency from January 1999 to December 2001, Zhang (2008) estimates a three-factor credit default swap model and computes the implied one-year physical and risk-neutral default probabilities. In line with our results, his risk-neutral default probability is always higher than its physical counterpart, and the wedge between them is time-varying, and typically increases with the physical default probability.

5.5 Robustness Check

Different degrees of concern about model misspecification. In Table 4 we report some business cycle statistics from the simulations of our model for different degrees of model uncertainty and no risk aversion on the lenders’ side. We start with no fears about model misspecification, i.e. $\theta = +\infty$, and we lower the penalty parameter to 0.25, for which we obtain a detection error probability of 9.5 percent. As expected when we reduce the value of $\theta$, we observe that the frequency of default goes down. To keep it at the historical level of 3 percent per year, we make the borrower more impatient by adjusting $\beta$ downwards.

In the comparison across models, which typically differ in several dimensions along their parametrization and assumed functional forms, it may be hard to identify which key ingredient is driving each difference in the simulated statistics. Table 4 helps us highlight the contribution of model uncertainty by showing how the dynamics of relevant macro variables vary in the same environment as we increase the preference for robustness.

40
The first feature that stands out is that both the mean and the standard deviation of bond spreads increase with the lenders’ concerns about model misspecification. For the same default frequency, as $\theta$ decreases, a greater degree of concern about model misspecification tends to push up the probability distortions associated with low utility states for the lender, in particular those states in which default occurs.

Note, however, that on average borrowing almost does not decline as its cost goes up. While this is true for long-term debt, it is not when the borrower can only issue one-period bonds. In the latter case, the borrower adjusts much more its debt level as output slides down and default risk (under both models) increases. This follows from the fact that, as explained before, the disincentives to issue an additional bond are larger with one-period bonds that with long-term debt.\(^{57}\)

Given that borrowing does not adjust enough to compensate for prices variations, interest repayments become more volatile as $\theta$ decreases. Consequently, we observe more variability in trade balance and consumption relative to output. Consistently, the opposite occurs with one-period debt.

### Table 4: Business Cycle Statistics for Different Degrees of Robustness

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$\theta = +\infty$</th>
<th>$\theta = 5$</th>
<th>$\theta = 1$</th>
<th>$\theta = 0.75$</th>
<th>$\theta = 0.5$</th>
<th>$\theta = 0.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean($r - r^f$)</td>
<td>4.54</td>
<td>4.83</td>
<td>6.43</td>
<td>7.23</td>
<td>9.30</td>
<td>18.74</td>
</tr>
<tr>
<td>Std.dev.($r - r^f$)</td>
<td>3.32</td>
<td>3.41</td>
<td>3.96</td>
<td>4.28</td>
<td>5.10</td>
<td>9.36</td>
</tr>
<tr>
<td>Mean($-b/y$)</td>
<td>43.32</td>
<td>43.40</td>
<td>43.93</td>
<td>43.96</td>
<td>43.78</td>
<td>42.22</td>
</tr>
<tr>
<td>Std.dev.(c)/std.dev.(y)</td>
<td>1.17</td>
<td>1.18</td>
<td>1.21</td>
<td>1.22</td>
<td>1.23</td>
<td>1.22</td>
</tr>
<tr>
<td>Std.dev.($tb/y$)</td>
<td>0.86</td>
<td>0.92</td>
<td>1.11</td>
<td>1.17</td>
<td>1.28</td>
<td>1.40</td>
</tr>
<tr>
<td>Corr($y, c$)</td>
<td>0.99</td>
<td>0.99</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.97</td>
</tr>
<tr>
<td>Corr($y, r - r^f$)</td>
<td>$-0.79$</td>
<td>$-0.78$</td>
<td>$-0.77$</td>
<td>$-0.76$</td>
<td>$-0.74$</td>
<td>$-0.67$</td>
</tr>
<tr>
<td>Corr($y, tb/y$)</td>
<td>$-0.77$</td>
<td>$-0.76$</td>
<td>$-0.72$</td>
<td>$-0.70$</td>
<td>$-0.66$</td>
<td>$-0.55$</td>
</tr>
<tr>
<td>DEP</td>
<td>0.50</td>
<td>0.469</td>
<td>0.377</td>
<td>0.344</td>
<td>0.267</td>
<td>0.095</td>
</tr>
<tr>
<td>Default frequency (annually)</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
</tr>
</tbody>
</table>

57 In the working paper version, Pouzo and Presno (2012) report the MC statistics for different degrees of concern about model misspecification for one-period debt.

### 6 Micro-foundations for ad-hoc Pricing Kernels

In recent years, several studies on quantitative sovereign default models have considered ad-hoc pricing kernels to improve the calibration along the asset-pricing dimension while keeping the model tractable and easy to solve. Some examples include Arellano (2008), Arellano and Ramanarayanan (2012), and Hatchondo et al. (2010). We view our model as providing foundations for this class of
Statistic  | Data  | Baseline Model  | Our Model  \\
--- | --- | --- | ---
Mean\((r - r_f)\) | 8.15  | 3.47 (1.9,4.8) | 8.18 (4.5,13.7) \\
Std.dev.\((r - r_f)\) | 4.58  | 2.92 (1.9,3.9) | 4.57 (2.4,7.6) \\
mean\((-b/y)\) | 46 | 43 (22,67) | 43 (22,68) \\
Std.dev.\((c)/\text{std. dev.} \ (y)\) | 0.87  | 1.96 | 2.08 \\
Std.dev.\((tb/y)\) | 1.21  | 5.07 (3.5,6.6) | 5.84 (4.4,7.4) \\
Corr\((y,c)\) | 0.97  | 0.76 (0.7,0.8) | 0.70 (0.6,0.8) \\
Corr\((y,r-r_f)\) | -0.72 | -0.52 (-0.8,-0.08) | -0.72 (-0.8,-0.6) \\
Corr\((y,tb/y)\) | -0.77 | -0.36 (-0.6,-0.09) | -0.31 (-0.5,-0.1) \\
Drop in \(y\) (around default) | -6.4 | -7.1 (11.5,4.0) | -9.3 (14.6,4.5) \\
DEP | NA | 0.50 | 0.247 \\
Default frequency (annually) | 3.00 | 3.00 (1.7,4.4) | 3.00 (1.6,4.6) \\

Table 5: Business Cycle Statistics for our Model and Baseline Model with One-period Bonds, and the Data. The 90-percent confidence interval generated by the MC simulations are reported in parenthesis.

ad-hoc pricing kernels. In this section, we study the differences and similarities between the our and the ad-hoc pricing kernels, both theoretically and quantitatively.

In the aforementioned papers, the pricing kernel is given by an ad-hoc function that belongs to the class \(S\) defined by

\[
S \equiv \{S: \mathbb{Y} \times \mathbb{Y} \to \mathbb{R}_{++} \text{ such that } E[S(y_t,Y_{t+1})|y_t] = \gamma \text{ and } S(y_t,\cdot) \text{ is non-increasing}\},
\]

where \(\gamma\) is the lenders’ time discount factor, which is equal to the reciprocal of the gross risk-free rate, i.e. \(\gamma = \frac{1}{1+r_f}\). Note that \(S(y_t,\cdot)\) scaled up by \(\frac{1}{\gamma}\), i.e. \(\frac{S(y_t,\cdot)}{\gamma}\), is a pdf on \(\mathbb{Y}\). In what follows we assume that \(Y\) has a pdf, denoted by \(f_{Y\mid Y}\), and that the pdf embedded in \(S(y_t,\cdot)\) and \(f_{Y\mid Y}\) are equivalent.\(^{58}\)

A common example is

\[
S(y_t,y_{t+1}) = \gamma \exp\{-\eta v_{t+1} - 0.5(\eta \sigma_v)^2\},
\]

where \(\eta > 0, v_{t+1} \sim N(0,\sigma_v^2)\), and the endowment of the borrower follows an AR(1),

\[
\log y_{t+1} = \alpha + \rho \log y_t + v_{t+1}.
\]

This process is typically assume in the literature (it is used in our simulation results as well) and facilitates the exposition.

\(^{58}\)Two probability measures are equivalent if they are absolutely continuous with respect to each other.
It is easy to see that the equilibrium price function associated to an ad-hoc pricing kernel \(S \in \mathcal{S}\) and an arbitrary stochastic process for output with pdf \(f_{Y'|Y}\) is given by

\[
B' \mapsto q_a(y, B') = E_Y[\{\lambda + (1 - \lambda)(\psi + q_a(y', B^*_a(y', B'))\}]\delta_a(y', B')S(y, y')|y],
\]

for any \((y, B') \in \mathcal{Y} \times \mathbb{B};\) where \(E_Y[\cdot|y]\) is computed under pdf \(f_{Y'|Y},\) and \(B^*_a\) and \(\delta_a\) denote the equilibrium debt and default policies, respectively, given pricing kernel \(S.\)

Due to the equivalence assumption, it is easy to see that

\[
q_a(y, B') = \gamma \int_{\mathcal{Y}} \{\lambda + (1 - \lambda)(\psi + q_a(y', B^*_a(y', B'))\}]\delta_a(y', B')\varphi(y'|y)dy',
\]

where \(\varphi(\cdot|y)\) is a new pdf that depends on the primitives of pdf \(f_{Y'|Y}\) and parameters of \(S.\) That is, using this ad-hoc pricing kernel is equivalent to using a modified version of the conditional probability governing the stochastic process of the endowment. Moreover, for any \(S \in \mathcal{S},\) \(\varphi(\cdot|y)\) is first order dominated by \(f_{Y'|Y}(\cdot|y).\)

We view this as a noteworthy similarity with our model pricing kernel, \(\gamma_{m_R},\) which also results on a pricing equation that uses a distorted version of the conditional probability governing the stochastic process of the endowment. Our model pricing kernel, however, emerges endogenously in general equilibrium from the lenders’ attitude towards model uncertainty, and this fact has important consequences. First, our conditional distorted probability is not Markov, i.e., depends on the entire past history (as opposed to only depending on last period value) of endowment. This is due to the fact that our conditional distorted probability depends on \(B^*_a\) (and access to financial markets), whereas the probability measure in equation (23) does not.

We finally observe that for our previous example with the kernel specification (21) and output process (22), \(\varphi(\cdot|y)\) is a log-normal pdf with parameters \((-\sigma^2 + \alpha + \rho \log y, \sigma^2),\) i.e. \(\varphi(\cdot|y) = \frac{\exp \left( \frac{(\alpha + \rho \log y)^2}{2 \sigma^2} \right)}{\sqrt{2\pi \sigma^2}}\)

Define \(\tilde{S} : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_{++}\) such that \(S(y, y') = \frac{\tilde{S}(y, y')}{E_Y[S(y, Y')]}\). Thus, \(\phi(y'|y)\) can be written as \(\frac{\tilde{S}(y, y')}{E_Y[S(y, Y')]}f_{Y'|Y}(y'|y).\) By assumption, \(\tilde{S}(y, \cdot)\) is nonincreasing, and, thus,

\[
\int_{-\infty}^{t} \varphi(y'|y)dy' \geq \int_{-\infty}^{t} \frac{F_{Y'|Y}(t|y)}{E_Y[S(y, Y')]} \int_{-\infty}^{\infty} \tilde{S}(y, y') \left\{ y' \leq t \right\} f_{Y'|Y}(y'|y)dy'.
\]

because \(\left\{ \frac{F_{Y'|Y}(\cdot|y)}{E_Y[S(y, Y')]} : t \in \mathbb{R} \right\} \) is a family of pdf that is decreasing (in a FOSD sense) with respect to \(t.\)

The term in the second line equals \(\tilde{F}_{Y'|Y}(t|y)\) and thus the result follows.
\(\phi(\cdot; -\sigma^2 \eta + \alpha + \rho \log y, \sigma^2)\).\(^60\) In particular, the conditional probability used in the pricing equation is still \textit{log-normal with the same variance but with lower conditional mean}; that is, it is first order dominated by the one governing the stochastic process of the endowment, and the parameter \(\eta\) regulates how different these two distributions are. Observe, however, that even with the output process (22), the conditional distorted probability measure in our model is not longer log-normal; in particular, it is skewed to the left, as shown in Figure 1.\(^61\)

In order to shed further light on asset-pricing implications of the ad-hoc pricing kernel and our pricing kernel, we find convenient to work with the modified pdf and the distorted pdf and to assume that the default set is of the threshold type and the same across different pricing kernels. We also focus the analysis on the short-term debt model, i.e., \(\lambda = 1\). The assumption over the default sets being of the the same and of the threshold type, although is ad-hoc, it seems to hold true in the numerical simulations and also have been shown to hold in different environments for these type of models; see Arellano (2008) and Pouzo (2010).

Formally, let \(i \in \{\eta, \theta\}\) where \(\eta, \theta\) denotes the economy with ad-hoc (ours) pricing kernel. Suppose the stochastic process for the endowment is given by equation (22), \(\lambda = 1\) and \(B' \mapsto D^*_t(B') = D^*(B') \equiv \{y': y' \leq \bar{y}(B')\}\), then, for all \(B'\), the spread can be constructed as follows:

\[
\text{Sp}_{i,t+1}(B') = \left(\gamma \int_{y' > \bar{y}(B')} f^i_{t+1}(y'|y^{t+1})\right)^{-1} - \gamma^{-1} = \gamma^{-1} \left(\frac{1}{1-F^i_{t+1}(\bar{y}(B')|y^{t+1})} - 1\right) = \gamma^{-1} \frac{F^i_{t+1}(\bar{y}(B')|y^{t+1})}{1-F^i_{t+1}(\bar{y}(B')|y^{t+1})},
\]

where \(f^i(\cdot|y^t)\) (\(F^i(\cdot|y^t)\)) is the conditional pdf (cdf) of the model \(i\) given history \(y^t\).

It follows that, if for a given \(y_t\),

\[F^\eta_t(\cdot|y^t) > (<) F^\theta_t(\cdot|y^t)\] (24)

holds, then \(\text{Sp}_{\eta,t+1}(B') > (<) \text{Sp}_{\theta,t+1}(B')\) a.s.. Equation (24) essentially gives necessary and sufficient conditions that rank the the distance

\(^{60}\)The function \(y \mapsto \phi(y; \mu, \sigma^2)\) denotes a log-Gaussian pdf with parameters \((\mu, \sigma^2)\), i.e.:

\[y \mapsto \phi(y; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma y}} \exp\{-0.5(\log y - \mu)^2\}.\]

\(^{61}\)Surprisingly, this modified pdf resembles the distorted pdf that emerges endogenously under model uncertainty in Barillas et al. (2009) to analyze the equity premium and the risk-free rate puzzle in the context of Hansen and Jagannathan (1991) bounds. Both for a random walk process and a trend stationary process for log consumption, the distorted pdf results as well from a conditional mean shift in the approximating one.
between the conditional probability induced by the ad-hoc pricing kernel and our distorted measure. It states that if, say, the "slanted" probability used for pricing with our pricing kernel is first order dominated by the modified conditional probability associated to the ad-hoc pricing kernel, then spreads in our model are relatively higher. Whether one pdf is first order dominated by the other one, hinges on the specific parametrization used in the model and in the ad-hoc function $S$.

A few remarks regarding this result are in order. First, observe that for states with high values of endowment, as shown in the bottom panels of Figure 2, our conditional distorted pdf is well-approximated by $F_{\theta_{t}\mid Y}$— i.e., distortions are negligible —, consequently, we expect equation (24) to hold with the “<” inequality; i.e., our conditional cdf $F_{\theta_{t}}$ dominates (in first order stochastic sense) the cdf corresponding to $F_{\eta_{t}}$. We then conclude that for these states, our model generates an spread that is lower than then one generated by the model with ad-hoc pricing kernel. On the other hand, for states with low endowments, we expect our distorted conditional probability measure to put more weight on low values of future endowment than $F_{\eta_{t}}$; e.g. see the top panels of Figure 2, so the inequality in equation should be reversed and therefore our model generates an spread that is higher than then one generated by the model with ad-hoc pricing kernel.

Finally, we explore the quantitative implications of our pricing kernel and the benchmark ad-hoc pricing kernel by computing some key statistics of the spreads. To do so, we calibrate the ad-hoc pricing kernel using formula (21) by targeting the average level of spreads. The results are reported in Table 6. While the average spreads are the same across models by construction, there is a quite considerable similarity in the non-targetted moments: the volatility of spreads and their comovement with output.

### 7 Detection Error Probabilities

In this section we use detection error probabilities (DEP) to measure the amount of model uncertainty in this economy and interpret the value of penalty parameter $\theta$ in the calibration. For this

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62 For this computation, we consider the same allocations for both pricing kernels which are the equilibrium outcome in our calibrated economy. The statistics are computed using MC simulations following the same approach as in Section 5.1.

63 In the calibration, the values for $\eta$ are 14.93 and 8.31 for one-period debt and long-term debt, respectively.
we follow Anderson et al. (2003), Maenhout (2004), and Barillas et al. (2009) among others. We also propose an alternative way of interpreting $\theta$.

The impact of the parameter $\theta$ on the probability distortions and other endogenous quantities of the model is environment specific. It depends on other parameters of the model and on the parametrization itself. However, from the Bellman equations in Section 2.6, it is easy to see that the higher $\theta$ is, the lower the concerns for model misspecification are, and in turn, the closer the distorted and approximating models are. As explained in Section 5.1, we use DEP to formalize this relationship, in line with Barillas et al. (2009).

Let $L_{A,T}$ and $L_{\theta,T}$ be the likelihood functions corresponding to the approximating and distorted models for $(Y_t)_{t=1}^T$, respectively. Let $\Pr(\cdot \mid L_A)$ and $\Pr(\cdot \mid L_\theta)$ be the respective probabilities over the data, generated under the approximating and distorted models. Let $p_{A,T}(\theta) \equiv \Pr\left(\log\left(\frac{L_{A,T}}{L_{\theta,T}}\right) > 0 \mid L_A\right)$ and $p_{D,T}(\theta) \equiv \Pr\left(\log\left(\frac{L_{A,T}}{L_{\theta,T}}\right) < 0 \mid L_\theta\right)$, let the DEP be obtained by averaging $p_{D,T}(\theta)$ and $p_{A,T}(\theta)$:

$$DEP_T(\theta) = \frac{1}{2} (p_{A,T}(\theta) + p_{D,T}(\theta)).$$

If the two models are very similar to each other, mistakes are likely, yielding high values of $p_A(\theta)$ and $p_D(\theta)$; the opposite is true if the models are not similar.\(^{64}\)

The aforementioned quantities can be approximated by means of simulation. We start by setting an initial debt level and endowment vector. We then simulate time series for output for $T' = 2,000 + T$ periods (quarters), where $T = 240$.\(^{65}\) The process is repeated 2,000 times.

For each time-series realization, we construct $L_{A,T}$ and $L_{\theta,T}$. We then compute the percentage of times the likelihood ratio test indicates that the worst-case model generated the data (when the data were generated by the approximating model); we denote this as $p_{A,T}(\theta)$.\(^{66}\) Similarly, we use $p_{D,T}(\theta)$ to denote the percentage of times the likelihood ratio test indicates that the approximating model generated the data (when the data were generated by the distorted model). Finally, we compute $DEP_T(\theta) \equiv \frac{1}{2} (p_{A,T}(\theta) + p_{D,T}(\theta))$.

For a given number of observations (in our case 74), as $\theta \to +\infty$, the approximating and distorted models become harder to distinguish from each other and the detection error probability converges to 0.5. If instead they are distant from each other, the detection error probability is

\(^{64}\)The weight of one-half is arbitrary; see Barillas et al. (2009) among others. Moreover, as the number of observations increases, the weight becomes less relevant, since the quantities $p_{A,T}$ and $p_{D,T}$ get closer to each other; as shown in Figure 5.

\(^{65}\)To make our results for the DEP comparable with those of Barillas et al. (2009) and Bidder and Smith (2011), we consider a similar number of periods and thereby $T = 240$ is chosen. If instead $T$ was set to replicate the number of periods used in the calibration, the DEP would be considerably higher for the same probability distortions. For both models, we ignore the first 2,000 observations in order to avoid any dependence on our initial levels of debt and endowments.

\(^{66}\)In the case of $L_A = L_\theta$ we count this as a false rejection with probability 0.5.
below 0.5, getting closer to 0 as the discrepancy between the models gets larger.

Following Barillas et al. (2009) we consider a threshold for the DEP of 0.2; values of $DEP_T(\theta)$ that are larger or equal are deemed acceptable. In our calibration, our DEP is above this threshold, since $DEP_T(\theta) = 0.313$. For this value of $\theta$ (and other parameters), $p_{A,T} = 0.306$ and $p_{D,T} = 0.321$. So the weight of 0.5 does not play an important role.

We conclude the section by proposing an alternative view for interpreting $\theta$. This is based on the following observation: for any fix finite $\theta$ (for which $L_{T}\theta,T$ exists), $L_{A,T} \neq L_{T}\theta,T$ with positive probability; thus, as the number of observations increases, $p_{T,k}(\theta) \rightarrow 0$ for $k = \{A,D\}$. Therefore, for a given level of $\theta$ and an a priori chosen level $\alpha \in (0,1)$, which does not depend on $\theta$, we can define $T_{\alpha,\theta} \equiv \max\{T; p_{T}(\theta) = \alpha\}$, as the maximum number of observations before $DEP_T(\theta)$ falls below $\alpha$. A heuristic interpretation of this number is that the agents need at least $T_{\alpha,\theta}$ observations to be able to distinguish between the two models at a certainty level of $\alpha$. The higher this number, the harder it is to distinguish between the models.

Figure 5 plots $\{p_{T,A}(\theta), p_{T,D}(\theta), DEP_T(\theta)\}_{T=90}^{2400}$ for the given value of $\theta$ in our calibration. For a level of $\alpha = 0.2$, we see that $T_{\alpha,\theta} \approx 700$. That is, one needs approximately 9.5 times our sample of 74, in order to obtain a level of $\alpha = 0.2$ for $DEP_T(\theta)$ and consequently claim that these models are sufficiently different from each other, according to this criterion.

Under both interpretations—the one using the threshold of 0.2 in Barillas et al. (2009) or looking at $T_{\alpha,\theta}$—the probability distortions associated with this value of $\theta$ are typically small. This
is result of the fact that the discrepancy between the approximating and distorted models is largest for values of the state where default occurs; see Figure 1. Since default is a “rare event” (e.g., in our calibration, it occurs approximately three times every 100 years), most of the time the discrepancy between the the approximating and distorted models is small.

8 Conclusion

This paper addresses a well-known puzzle in the sovereign default literature: why are bond spreads for emerging economies so high if default episodes are rare events, with a low probability of occurrence? Using Eaton and Gersovitz (1981) general equilibrium framework, extended by Chatterjee and Eyingunгор (2012) to allow for long-term debt, we provide an explanation to resolve this puzzle based on concerns about model misspecification.

International investors share an approximating or reference probability model for the underlying state of the economy which is their best estimate of the economy’s dynamics. They, however, distrust its specification due to scarce official reliable economic data and express their doubts by contemplating a set of alternative probability models which are statistical perturbations of the reference model. To guard themselves against specification errors, they seek decision rules that perform well across this set of models. To compensate for a risk and uncertainty-adjusted probability of default, they demand higher returns on their bond holdings. Thereby, our calibrated model can account for the bond spreads dynamics observed in the data while preserving the default frequency at the historical low levels and replicating, at the same time, the standard empirical regularities for emerging economies.

In recent years, some emerging economies such as Argentina, South Africa, Brazil, Colombia, and Turkey have issued GDP-indexed or inflation-indexed sovereign bonds. Credibility of the sovereign and transparency of its statistic agency are paramount for the success of these markets. Potential concerns of misreporting output growth or inflation would be priced in by investors when lending to the sovereign, and might question the desirability of these policies. Any discrepancy between the reference official rate and the market-perceived rate could constitute a form of partial default on the debt contract. While our model does not allow for this class of partial default events, we believe that our environment is sufficiently rich to be extended in a way to explore the pricing implications of these other financial instruments, especially in emerging economies.

Another interesting avenue for future work is to extend the results in Subection 4.2 and use our stochastic discount factor to price domestically traded and internationally traded assets. This will impose further restrictions in order to identify the value for $θ$ that go above and beyond the ones imposed here using detection error probabilities.
References


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A Moments of approximating and distorted densities.

Table 7 presents the computed moments for the approximating and distorted conditional densities of next-period $y_{t+1}$, given current $y_t$ and bond holdings $B_t$. As shown in Figure 1, the current endowment level $y_t$ is set to half a standard deviation below its unconditional mean, and the bond holdings $B_t$ is given by the median of its unconditional distribution in the simulations.

<table>
<thead>
<tr>
<th>Moment</th>
<th>Approximating Model</th>
<th>Distorted Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean($y_{t+1}$)</td>
<td>0.9518</td>
<td>0.9481</td>
</tr>
<tr>
<td>Std.dev.($y_{t+1}$)</td>
<td>0.0191</td>
<td>0.0202</td>
</tr>
<tr>
<td>Skewness($y_{t+1}$)</td>
<td>0.0601</td>
<td>0.0811</td>
</tr>
<tr>
<td>Kurtosis($y_{t+1}$)</td>
<td>3.0064</td>
<td>2.7910</td>
</tr>
</tbody>
</table>

Table 7: Moments for the approximating and distorted conditional densities.

By Law of Large Numbers, the moments for the approximating model are essentially the same to the corresponding “population” moments of the lognormal distribution. Regarding the distorted model, several moments differ significantly from those of the approximating model. In particular, there is a clear shift to the left of the conditional mean. Because of it, the skewness is higher, even though the distorted model puts more probability mass on low realizations of output, $y_{t+1}$, where default is optimal for the borrower, as illustrated in Figure 1.

B Robustness Check: Risk aversion with time-additive, standard expected utility.

As displayed in Table 8, plausible degrees of risk aversion on the lenders’ side with standard time separable expected utility, are not enough to generate sufficiently high bond spreads while keeping the default frequency as observed in the data.

We considered an exogenous stochastic process for the lenders’ consumption given by

$$\ln C_{t+1} = \rho^L \ln C_t^L + \sigma^L \varepsilon_{t+1},$$

where $\varepsilon_{t+1} \sim i.i.d. N(0,1)$. Shocks $\varepsilon_{t+1}$ and $\varepsilon_{t+1}^L$ are assumed to be independent. We estimate the log-normal AR(1) process for $C_t^L$ using U.S. consumption data.\(^{67}\)

Table 8 displays the business cycle statistics for different values of the lenders’ coefficient of relative risk aversion, $\sigma^L$, ranging from 1 to 50, and no fears about model misspecification, i.e. $\theta = +\infty$.\(^{68}\) First, as we would expect, bond spreads increase on average and become more volatile with the value of $\sigma^L$. They do so, however, to a very limited extent. Plausible degrees of risk aversion are not even close to generate sufficiently high bond spreads while keeping the default

\(^{67}\) Time series for seasonally adjusted real consumption of nondurables and services at a quarterly frequency are taken from the Bureau of Economic Analysis, in logs, and filtered with a linear trend. The estimates for parameters $\rho^L$ and $\sigma^L$ are 0.967 and 0.025, respectively.

\(^{68}\) For each value of $\sigma^L$, the discount factor for the borrower, $\beta$, is calibrated to replicate a default frequency of 3 percent annually.
frequency as observed in the data. Setting $\sigma^L$ equal to 50 generates average bond spreads of just 3.76 percent, less than half the value observed in the data. This high value for the coefficient of risk aversion is sufficient to explain the equity premium puzzle in Mehra and Prescott (1985). In contrast with the economy considered there, the stochastic discount factor in our model would not typically vary inversely with the bond payoff, limiting the ability of the model to generate sufficiently high bond spreads. Second, given a stationary process for consumption in our model, the net risk-free rate decreases and turns negative for sufficiently high values of $\sigma^L$, while its volatility grows dramatically. Facing lower risk-free rates, the borrower reacts by borrowing more. The variations in the debt-to-output ratio are, however, small, as in the case with model uncertainty. Finally, larger and more volatile capital outflows for interest payments translate into higher variability of consumption and net exports.

### C Calibration of One-Period Debt Model

In Table 9 we report the parameter values used to calibrate the model with one-period bonds. The value for the parameter $h$, controlling the variance of the preference shock, was gradually reduced to 0.0001. For this value no default in our simulations is driven by the preference shock.

### D Recursive Formulation of the Problem of the “Minimizing Agent”

In this section, we show that the principle of optimality holds for the Problem of the “minimizing agent”.

Let $c^L$ be a consumption plan. A feasible consumption plan is one that satisfies the budget constraint for each $t$. Preferences over consumption plans for lenders are then described as follows. For any given consumption plan $c^L$ and initial state $w_0$, the lifetime utility over such plan is given

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**Table 8: Business Cycle Statistics for Different Degrees of Risk Aversion.**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$\sigma^L = 1$</th>
<th>$\sigma^L = 2$</th>
<th>$\sigma^L = 5$</th>
<th>$\sigma^L = 10$</th>
<th>$\sigma^L = 20$</th>
<th>$\sigma^L = 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean($r - r^f$)</td>
<td>3.44</td>
<td>3.48</td>
<td>3.48</td>
<td>3.49</td>
<td>3.49</td>
<td>3.76</td>
</tr>
<tr>
<td>Std.dev.($r - r^f$)</td>
<td>2.61</td>
<td>2.63</td>
<td>2.59</td>
<td>2.61</td>
<td>2.61</td>
<td>2.82</td>
</tr>
<tr>
<td>Mean($r^f$)</td>
<td>4.05</td>
<td>4.04</td>
<td>3.98</td>
<td>3.79</td>
<td>3.05</td>
<td>-1.38</td>
</tr>
<tr>
<td>Std.dev.($r^f$)</td>
<td>0.19</td>
<td>0.39</td>
<td>0.96</td>
<td>1.91</td>
<td>3.83</td>
<td>9.12</td>
</tr>
<tr>
<td>Mean($-b/y$)</td>
<td>75.49</td>
<td>74.93</td>
<td>74.51</td>
<td>74.84</td>
<td>75.36</td>
<td>77.46</td>
</tr>
<tr>
<td>Std.dev.($c$)/std.dev.($y$)</td>
<td>2.32</td>
<td>2.31</td>
<td>2.31</td>
<td>2.33</td>
<td>2.36</td>
<td>2.66</td>
</tr>
<tr>
<td>Std.dev.($tb/y$)</td>
<td>6.65</td>
<td>6.64</td>
<td>6.63</td>
<td>6.69</td>
<td>6.85</td>
<td>7.96</td>
</tr>
<tr>
<td>Corr($y, c$)</td>
<td>0.70</td>
<td>0.70</td>
<td>0.70</td>
<td>0.70</td>
<td>0.69</td>
<td>0.64</td>
</tr>
<tr>
<td>Corr($y, r - r^f$)</td>
<td>-0.60</td>
<td>-0.61</td>
<td>-0.61</td>
<td>-0.60</td>
<td>-0.61</td>
<td>-0.57</td>
</tr>
<tr>
<td>Corr($y, tb/y$)</td>
<td>-0.36</td>
<td>-0.36</td>
<td>-0.36</td>
<td>-0.36</td>
<td>-0.36</td>
<td>-0.33</td>
</tr>
<tr>
<td>Default frequency (annually)</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
</tr>
</tbody>
</table>
Table 9: Parameter Values for One-period Bond Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Borrower</strong></td>
<td></td>
</tr>
<tr>
<td>Risk aversion</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>Time discount factor</td>
<td>$\beta$</td>
</tr>
<tr>
<td>Probability of reentry</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Output cost parameter $\kappa_1$</td>
<td></td>
</tr>
<tr>
<td>Output cost parameter $\kappa_2$</td>
<td></td>
</tr>
<tr>
<td>AR(1) coefficient for $y_t$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>Std. deviation of $\varepsilon_t$</td>
<td>$\sigma_{\varepsilon}$</td>
</tr>
<tr>
<td>Preference shock parameter $h$</td>
<td></td>
</tr>
<tr>
<td><strong>Lender</strong></td>
<td></td>
</tr>
<tr>
<td>Robustness parameter</td>
<td>$\theta$</td>
</tr>
<tr>
<td>Constant for $z$</td>
<td>$\log(\bar{z})$</td>
</tr>
<tr>
<td><strong>Bond</strong></td>
<td></td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>$r_f$</td>
</tr>
<tr>
<td>Decay rate</td>
<td>$\lambda$</td>
</tr>
</tbody>
</table>

by $^{69}$

$$U_0(c^L; w_0) \equiv \min_{(m_{t+1})_t} \sum_{t=0}^{\infty} \gamma^t E \left[ M_t(W^t) \left( c^L_t(W^t) + \theta \gamma E[m_{t+1}(-|W^t|)(Y_t)] \right) \mid w_0 \right]$$

$$E_Y[m_{t+1}(Y_{t+1}|W^t) \mid y_t] = 1,$$

where $E$ denotes the expectation with respect to $W^t$ under the probability measure $P$, $\gamma \in (0,1)$ is the discount factor, the parameter $\theta \in [\theta, +\infty]$ is a penalty parameter that measures the degree of concern about model misspecification, and the mapping $\mathcal{E} : \mathcal{M} \rightarrow L^\infty(\mathcal{Y})$, with $\mathcal{M}$ defined in Subsection 2.5, is the conditional relative entropy, given by (3).

We note that, since $B$ is bounded, and, in equilibrium, $q_t \in [0, \gamma]$; any feasible consumption plan is bounded, i.e., $|c^L_t(W^t)| \leq C < \infty$ a.s.

**Definition D.1.** Given a feasible consumption plan $c^L$, for each $(t, w^t)$, we say functions $(t, w^t, c^L) \mapsto U_t(c^L; w^t)$, satisfy the sequential problem of the “minimizing agent” (SP-MA) iff $^{70}$

$$U_t(c^L; w^t) = \min_{(m_{t+1})_t} \sum_{j=0}^{\infty} \gamma^j E \left[ \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) \left( c^L_{t+j}(W^{t+j}) + \theta \gamma E[m_{t+1}(-|W^{t+j}|)(Y_{t+j})] \right) \mid w_t \right],$$

$$E_Y[m_{t+1}(Y_{t+1}|W^t) \mid y_t] = 1,$$

$^{69}$Without the i.i.d. component $x_t$, the lifetime utility for the lender over $c^L$ would simply be given by

$$U_0(c^L; y_0) \equiv \min_{(m_{t+1})_t} \sum_{t=0}^{\infty} \gamma^t E \left[ M_t(Y^t) \left( c^L_t(Y^t) + \theta \gamma E[m_{t+1}(|Y^t|)(Y_t)] \right) \mid y_0 \right].$$

$^{70}$Note that, since $c^L_t \geq -K_0$ and $\theta \mathcal{E} \geq 0$, the RHS of the equation is always well defined in $[-K_0, \infty]$ where $K_0$ is some finite constant.
where $M_t \equiv \prod_{\tau=1}^{t} m_{\tau}$, $M_0 = 1$ and $E \left[ \cdot | w^t \right]$ is the conditional expectation over all histories $W^\infty$, given that $W^t = w^t$.

**Definition D.2.** Given a feasible consumption plan $c^L$, for each $(t, y^t)$, we say functions $(t, w^t, c^L) \mapsto U_t(c^L; w^t)$ satisfy the functional problem of the “minimizing agent” (FP-MA) iff

$$U_t(c^L; w^t) = c^L_t(w^t) + \gamma \min_{m_{t+1}(\cdot | w^t) \in M} \left\{ E_Y[m_{t+1}(Y_{t+1} | w^t)] U_{t+1}(c^L; w^t, Y_{t+1}) \mid y^t \right\} + \theta E[m_{t+1}(\cdot | w^t)](y^t),$$

(28)

where

$$E_Y[m_{t+1}(Y_{t+1} | w^t)] U_{t+1}(c^L; w^t, Y_{t+1}) \mid y^t \right\}.$$

Henceforth, we assume that in both definitions, the “min” is in fact achieved. If not, the definition and proofs can be modified by using “inf” at a cost of making them more cumbersome. We also assume that $U_t(c^L; \cdot)$, defined by (27), is measurable with respect to $W^\infty$.

**Theorem D.1.** For any feasible consumption plan $c^L$,

(a) If $(U_t(c^L; w^t))_{t,w^t}$ satisfies the SP-MA, then it satisfies the FP-MA.

(b) Suppose there exist a function $(t, w^t) \mapsto \bar{U}_t(c^L; w^t)$ that satisfy the FP-MA and

$$\lim_{T \to \infty} \gamma^{T+1} E \left[ M_{T+1}(w^{T+1}) \bar{U}_{T+1}(c^L; w^{T+1}) \mid w_0 \right] = 0,$$

(29)

for all $M_{T+1}$ such that $M_{T+1} = m_{T+1} M_T$, $M_0 = 1$ and $m_{t+1} \in M$. Then $(\bar{U}_t(c^L; w^t))_{t,w^t}$ satisfy the SP-MA.

The importance of this theorem is that it suffices to study the functional equation (28). The proof of the theorem requires the following lemma (the proof is relegated to the end of the section).

**Lemma D.1.** In the program 27, it suffices to perform the minimization over $(m_t)_t \in M$ where

$$M \equiv \left\{ (m_t)_t : m_t \in M \cap \sum_{j=0}^{\infty} \gamma^j E \left[ \frac{M_{t+j}(w^{t+j})}{M_t(w^t)} \right] E[m_{t+j+1}(\cdot | W^{t+j})](Y_{t+j}) \mid w^t \right\} \leq C_{C, \gamma, \theta} \forall y^t \right\},$$

where $C_{C, \gamma, \theta} = 2 \frac{C}{(1-\gamma)^2 \gamma^2}$.

**Proof of Theorem D.1.** Throughout the proof we use $E_Y|X$ to denote the expectation of random variable $Y$, given $X$. 

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(a) From the definition of SP-MA and equation (26), it follows that
\[ U_0(c^L; w_0) = \min_{(m_{t+1})} \{ c^L_0(w_0) + \theta \gamma E[m_1(\cdot|w_0)](y_0) \} \]
\[ + \gamma \sum_{t=1}^{\infty} \gamma^{t-1} E_{W^1|W_0} \left[ M_1(W^1) \left( E_{W^1|W_1} \left[ \frac{M_t(W_t)}{M_1(W_1)} \{ c^L_t(W^t) + \theta \gamma E[m_{t+1}(\cdot|W^t)](Y_t) \} | W_1 \right] \right) \right] | w_0 \}
\[ = \min_{(m_{t+1})} \{ c^L_0(w_0) + \theta \gamma E[m_1(\cdot|w_0)](y_0) \} \]
\[ \gamma E_{W^1|W_0} \left[ m_1(W^1) \left( \sum_{s=0}^{\infty} \gamma^s E_{W^s+1|W_1} \left[ \frac{M_{s+1}(W^{s+1})}{M_1(W_1)} \{ c^L_{s+1}(W^{s+1}) \} \right] \right) \right] | w_0 \].
\[ \geq \min_{m_1} \{ c^L_0(w_0) + \theta \gamma E[m_1(\cdot|w_0)](y_0) \} + \gamma E_{W^1|W_0} \left[ m_1(Y^1|w_0) (U_1(c^L; w_0, W_1)) | w_0 \right] \]
\[ = \min_{m_1} \{ c^L_0(w_0) + \theta \gamma E[m_1(\cdot|w_0)](y_0) \} + \gamma E_{Y_1|Y_0} \left[ m_1(Y_1|w_0) E_X \left[ (U_{t+1}(c^L; w_t, X_{t+1}, Y_{t+1})) \right] \right] | y_0 \].

where the first inequality follows from definition of \( U \). The step (*) follows from interchanging the summation and integral (we show this fact towards the end of the current proof).

The final expression actually holds for any state \((t, y')\),
\[ U_t(c^L; w^t) \geq \min_{m_{t+1}} \{ c^L_t(w^t) + \theta \gamma E[m_{t+1}(\cdot|w^t)](y_t) \}
\[ + \gamma E_{Y_{t+1}|Y_t} \left[ m_{t+1}(Y^{t+1}|w^t) E_X \left[ (U_{t+1}(c^L; w_t, X_{t+1}, Y_{t+1})) \right] \right] | y_t \} \].

(30)

On the other hand, by definition of \( U \),
\[ U_0(c^L; w_0) \leq M_0(w_0) \{ c^L_0(w_0) + \theta \gamma E[m_1(\cdot|w_0)](y_0) \}
\[ + \gamma \sum_{t=1}^{\infty} \gamma^{t-1} E_{W^1|W_0} \left[ M_1(W^1) \left( E_{W^1|W_1} \left[ \frac{M_t(W_t)}{M_1(W_1)} \{ c^L_t(W^t) + \theta \gamma E[m_{t+1}(\cdot|W^t)](Y_t) \} | W_1 \right] \right) \right] \right] | w_0 \],

for any \((M_t)_t\) that satisfies the restrictions imposed in the text. In particular, it holds for \((M_t)_t\) where \(m_1\) is left arbitrary and \((m_t)_{t \geq 2}\) is chosen as the optimal one. By following analogous steps to those before, it follows that
\[ U_0(c^L; w_0) \leq \{ c^L_0(w_0) + \theta \gamma E[m_1(\cdot|w_0)](y_0) \} + \gamma E_{Y_1|Y_0} \left[ m_1(Y_1|w_0) \left( E_X [U_1(c^L; w_0, X_1, Y_1)] \right) \right] | y_0 \],

for any \(m_1\) that satisfies the restrictions imposed in the text; it thus holds, in particular, for the value that attains the minimum. Note that this holds for any \((t, y')\), not just \((t = 0, y_0)\), i.e.,
\[ U_t(c^L; w^t) \leq \min_{m_{t+1}} \{ c^L_t(w^t) + \theta \gamma E[m_{t+1}(\cdot|w^t)](y_t) \}
\[ + \gamma E_{Y_{t+1}|Y_t} \left[ m_{t+1}(Y_{t+1}|w^t) \left( E_X \left[ U_{t+1}(c^L; w^t, X_{t+1}, Y_{t+1}) \right] \right) \right] | y_t \} \].

(31)

Therefore, putting together equations (30) and (31), it follows that \((U_t)_t\) satisfy the FP-MA.

(b) Let \((U_t)_t\) satisfy the FP-MA and equation (29). Then, by a simple iteration it is easy to
see that

\[
\tilde{U}_0(c^L; w_0) \leq \lim_{T \to \infty} \sum_{j=0}^{T} \gamma^j E_{W^j | W_0} \left[ (M_j(W^j)) \{c^L_j(W^j) + \theta \gamma E[m_{j+1}(|W^j)|(Y_j)] \} | w_0 \right] \\
+ \lim_{T \to \infty} \gamma^{T+1} E_{W^{T+1} | W_0} [M_{T+1}(W^{T+1}) (\tilde{U}_{T+1}(c^L; W^{T+1})) | w_0].
\]

The last term in the RHS is zero by equation (29), so \( \tilde{U}_0(c^L; w_0) \leq U_0(c^L; w_0) \) (where \( U \) satisfies the SP-MA). The reversed inequality follows from similar arguments and the fact that \( U_0(c; w_0) \) is the minimum possible value.

The proof for \((t, y^t)\) is analogous. Therefore, we conclude that any sequence of functions \((\tilde{U}_t)_t\) that satisfies FP-MA and (29), also satisfies SP-MA.

**Proof of \(\star\).** To show \(\star\) is valid, let

\[
H_n \equiv \sum_{s=0}^{n} m_1(y_1 | w_0) \gamma^s E_{W^{s+1} | W_1} \left[ \frac{M_{s+1}(W^{s+1})}{M_1(w^1)} \{c^L_{s+1}(W^{s+1}) + \theta \gamma E[m_{s+2}(|W^{s+1}|)(Y_{s+1})] \} | w^1 \right].
\]

We note that

\[
|H_n| \leq \sum_{s=0}^{\infty} m_1(y_1 | w_0) \gamma^s CE_{W^{s+1} | W_1} \left[ \frac{M_{s+1}(W^{s+1})}{M_1(w^1)} \right] | w^1 |
\]

\[
+ \sum_{s=0}^{\infty} m_1(y_1 | w_0) \gamma^s E_{W^{s+1} | W_1} \left[ \frac{M_{s+1}(W^{s+1})}{M_1(w^1)} \theta \gamma E[m_{s+2}(|W^{s+1}|)(Y_{s+1}) | w^1] \right],
\]

where the second line follows because \(c^L_t\) is bounded. Observe that, \( E_{W^{s+1} | W_1} \left[ \frac{M_{s+1}(W^{s+1})}{M_1(w^1)} \right] | w^t = 1 \) for all \(t\) and \(j\) and \(\sum_{s=0}^{\infty} \gamma^s E_{W^{s+1} | W_1} \left[ \frac{M_{s+1}(W^{s+1})}{M_1(w^1)} \theta \gamma E[m_{s+2}(|W^{s+1}|)(Y_{s+1}) | w^1] \right] \leq C_{C, \gamma, \theta} \) by Lemma D.1. Hence

\[
|H_n| \leq m_1 \times K_0
\]

for some \(\infty > K_0 > 0\) (it depends on \((\gamma, \theta, M)\)). Since the RHS is integrable, by the Dominated Convergence Theorem, interchanging summation and integration is valid.

**Proof of Lemma D.1.** Before showing the desired results, we show it suffices to perform the minimization over \((m_t)_t \in \mathcal{M}\) where

\[
\mathcal{M} \equiv \left\{ (m_t)_t: m_t \in \mathcal{M} \cap \sum_{j=0}^{\infty} \gamma^j E \left[ \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) E[m_{t+j+1}(|W^{t+j}|)(Y_{t+j})] | w^t \right] \leq C_{C, \gamma, \theta}, \forall y^t \right\},
\]

where \(C_{C, \gamma, \theta} = 2 \frac{C}{(1-\gamma)\theta^2}\).

We do this by contradiction. Suppose that \((m_t)_t\) solves the minimization problem in SP-MA, and,
\[ \sum_{j=0}^{\infty} \gamma^j E \left[ \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) \mathcal{E}[m_{t+j+1}(\cdot|W^{t+j})|(Y_{t+j})] | w^t \right] > C_{C, \gamma, \theta}. \]

Since consumption is bounded
\[ \mathcal{U}_t(c^L; w^t) = \sum_{j=0}^{\infty} \gamma^j \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) \left\{ c_{t+j}^L(W^{t+j}) + \gamma \theta \mathcal{E}[m_{t+j+1}(\cdot|W^{t+j})|(Y_{t+j})] | w^t \right\} \]
\[ \geq \sum_{j=0}^{\infty} \gamma^j \left\{ (-C) \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) | w^t \right\} + \theta \gamma E \left[ \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) \mathcal{E}[m_{t+j+1}(\cdot|W^{t+j})|(Y_{t+j})] | w^t \right]. \]

Note that
\[ E \left[ \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) | w^t \right] = \int_{\omega^{t+j-1}|w^t} \frac{M_{t+j-1}(\omega^{t+j-1})}{M_t(w^t)} \left\{ \int_{\omega} m_{t+j}(\omega^{t+j-1}y_{t+j-1}) P(dy_{t+j}|y_{t+j-1}) \right\} Pr(d\omega^{t+j-1}|w^t) \]
\[ = \int_{\omega^{t+j-1}|w^t} \frac{M_{t+j-1}(\omega^{t+j-1})}{M_t(w^t)} Pr(d\omega^{t+j-1}|w^t) = ... = 1. \]

where \( Pr \) is the conditional probability over histories \( \omega^{\infty} \), given \( W^t = w^t \). Hence,
\[ \sum_{j=0}^{\infty} \gamma^j E \left[ \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) \left\{ c_{t+j}^L(W^{t+j}) + \gamma \theta \mathcal{E}[m_{t+j+1}(\cdot|W^{t+j})|(Y_{t+j})] | w^t \right\} \right] \]
\[ \geq - \frac{C}{1-\gamma} + \theta \gamma E \left[ \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) \mathcal{E}[m_{t+j+1}(\cdot|W^{t+j})|(Y_{t+j})] | w^t \right]. \]

By assumption, the second term is larger than \( \theta \gamma C_{C, \gamma, \theta} \). Hence, the value for the minimizing agent of playing \( (m_t)_t \) is bounded below by \( -\frac{C}{1-\gamma} + \theta \gamma C_{C, \gamma, \theta} \). By our choice of \( C_{C, \gamma, \theta} \),
\[ -\frac{C}{1-\gamma} + \theta \gamma C_{C, \gamma, \theta} = \frac{C}{1-\gamma}. \]

Since \( \mathcal{E}[1] = 0 \), the RHS of the previous display is larger than
\[ \sum_{j=0}^{\infty} \gamma^j E \left[ \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) \left\{ c_{t+j}^L(W^{t+j}) + \theta \gamma \mathcal{E}[1](Y_{t+j}) \right\} | w^t \right]. \]

Therefore, we conclude that
\[ \mathcal{U}_t(c^L; w^t) > \sum_{j=0}^{\infty} \gamma^j E \left[ \left( \frac{M_{t+j}(W^{t+j})}{M_t(w^t)} \right) \left\{ c_{t+j}^L(W^{t+j}) + \theta \gamma \mathcal{E}[1](Y_{t+j}) \right\} | w^t \right]. \]

But since \( m_t = 1 \) for all \( t \) is a feasible choice, this is a contradiction to the definition of \( \mathcal{U}_t(c^L; w^t) \).

\[ \square \]

E  Proofs of Theorems 4.1 and 4.2

In what follows we describe in more detail the environment presented in Section 4, where a more general specification for the process of the lenders’ endowment is assumed and we allow the lender
to distrust it.

Analogously to Section 2.5, lenders’ utility over consumption plans $c_t$ after any history $(t, y^t, z^t)$ is henceforth given by

$$U_t(c_t^L; y^t, z^t) = c_t^L(y^t, z^t) + \gamma \min_{(m,n) \in \mathcal{M} \times \mathcal{N}} \left\{ \theta E_\theta [m](y_t) + E_Y [m(Y_{t+1})|\eta \mathcal{E}_{\eta}[n](z_{t}) \right\} + n(Z_{t+1})U_{t+1}(c_{t+1}; y_{t+1}, z_{t+1}) \big| y_t, z_t \big\},$$

where the conditional relative entropies $\mathcal{E}_\theta : \mathcal{M} \rightarrow \{ g: \mathcal{Y} \rightarrow \mathbb{R}_+ \}$ and $\mathcal{E}_\eta : \mathcal{N} \rightarrow \{ g: \mathcal{Z} \rightarrow \mathbb{R}_+ \}$ are defined in analogy to (3).

By similar calculations to those in Appendix D, one can show that the corresponding Bellman equation is

$$W_R(v_t, B_t, b_t) = \min_{(m,n) \in \mathcal{M} \times \mathcal{N}} \max_{b_{t+1}} \left\{ z_t + G(b_t, b_{t+1}; B_{t+1}, v_t) + \theta \gamma \mathcal{E}[m](y_t) + \eta \gamma E_Y [m(Y_{t+1})|\eta \mathcal{E}_{\eta}[n](z_{t})|y_t] \right\} + \gamma E_V [m(Y_{t+1})n(Z_{t+1})W(V_{t+1}, B_{t+1}, b_{t+1})|v_t] \big| y_t \big\},$$

where let $(z_t, b_t, b_{t+1}; B_{t+1}, v_t) \rightarrow z_t + G(b_t, b_{t+1}; B_{t+1}, v_t) \equiv z_t + q(v_t, B_{t+1})(b_{t+1} - (1 - \lambda)b_t) - (\lambda + (1 - \lambda)\psi)b_t$ be the per-period payoff. The expression for $W_A$ is analogous.

Proof of Theorem 4.1. Let $v_t \equiv (z_t, y_t)$. The proof consists of two parts. First, assuming that $q(v_t, B_{t+1}) = q(y_t, B_{t+1})$ we show that $W_i(v_t, B_t, b_t) = A_0 + A_1z_t + \bar{W}(y_t, B_t, b_t)$ for all $i \in \{ R, A \}$ where

$$A_0 = \frac{\rho_0 A_1}{1 - \gamma} \text{ and } A_1 = \frac{1}{1 - \gamma \rho_1}. \tag{32}$$

Then, given this result, we prove that the equilibrium price function from the FONC of the lender’s problem is in fact $q(v_t, B_{t+1}) = q(y_t, B_{t+1})$. This shows that the equilibrium mapping that maps prices into prices, in fact maps functions of $(y_t, B_{t+1})$ onto themselves, and thus the equilibrium price must have this property.

Observe that, given $q(v_t, B_{t+1}) = q(y_t, B_{t+1})$, the borrower does not consider $z_t$ as part of the state, and thus $\delta(v_{t+1}, B_{t+1}) = \delta(y_{t+1}, B_{t+1})$. Hence,

$$W(v_{t+1}, B_{t+1}, b_{t+1}) = \delta(y_{t+1}, B_{t+1})W_R(v_{t+1}, B_{t+1}, b_{t+1}) + (1 - \delta(y_{t+1}, B_{t+1}))W_A(v_{t+1})$$

$$= \delta(y_{t+1}, B_{t+1})W_R(v_{t+1}, B_{t+1}, b_{t+1}) + (1 - \delta(y_{t+1}, B_{t+1}))W_A(v_{t+1}) + A_0 + A_1z_{t+1}$$

$$\equiv \bar{W}(y_{t+1}, B_{t+1}, b_{t+1}) + A_0 + A_1z_{t+1}.$$

Given the assumption on prices, it follows that

$$W_R(v_t, B_t, b_t) = \min_{(m,n) \in \mathcal{M} \times \mathcal{N}} \max_{b_{t+1}} \left\{ z_t + G(b_t, b_{t+1}; B_{t+1}, y_t) + \theta \gamma \mathcal{E}[m](y_t) \right\} + \eta \gamma E_Y [m(Y_{t+1})|\eta \mathcal{E}_{\eta}[n](z_{t})|y_t] + \gamma E_V [m(Y_{t+1})n(Z_{t+1})W(V_{t+1}, B_{t+1}, b_{t+1})|v_t] \big| y_t \big\},$$

where $\mathcal{N}$ is defined similarly to $\mathcal{M}$.

Solving to the minimization problem yields

$$W_R(v_t, B_t, b_t) = \max_{b_{t+1}} \left\{ z_t + G(b_t, b_{t+1}; B_{t+1}, y_t) - \theta \gamma \log E_Y \left[ \exp \left\{ -\frac{\left[ W(\cdot, Y_{t+1}, B_{t+1}, b_{t+1})\right] (Y_{t+1}, z_{t})}{\theta} \right\} \right| y_t \right\}. \tag{32}$$

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By assumption over $W$, we have that
\[
W_R(y_t, B_t, b_t) = \bar{W}(y_t, B_t, b_t) + A_1(z_t) + A_0
\]
\[
= \max_{b_{t+1}} \left\{ G(b_t, b_{t+1}; B_{t+1}, y_t) - \theta \gamma \log E_Y \left[ \exp \left\{ -\frac{T_\eta \bar{W}(Y_{t+1}, B_{t+1}, b_{t+1}) + A_0 + A_1 Z_{t+1}(Y_{t+1}, y_t)}{\theta} \right\} | y_t \right] \right\} + z_t
\]
\[
= \max_{b_{t+1}} \left\{ G(b_t, b_{t+1}; B_{t+1}, y_t) - \theta \gamma \log E_Y \left[ \exp \left\{ -\frac{A_1 T_\eta/A_1[\epsilon_{t+1}](Y_{t+1}, y_t) + \bar{W}(Y_{t+1}, B_{t+1}, b_{t+1})}{\theta} \right\} | y_t \right] \right\} + z_t \gamma A_0
\]
\[
= \max_{b_{t+1}} \left\{ G(b_t, b_{t+1}; B_{t+1}, y_t) - \theta \gamma \log E_Y \left[ \exp \left\{ -\frac{A_1 T_\eta/A_1[\epsilon_{t+1}](Y_{t+1}, y_t) + \bar{W}(Y_{t+1}, B_{t+1}, b_{t+1})}{\theta} \right\} | y_t \right] \right\} + (\gamma A_1 \rho_1 + 1)(z_t) + \gamma(A_0 + A_1 \rho_0).
\]
Therefore, it must be the case that
\[
A_1 = (\gamma A_1 \rho_1 + 1) \text{ and } A_0 = \gamma(A_0 + A_1 \rho_0).
\]

Similar algebra for $W_A$ yields
\[
W_A(y_t) + A_0 + A_1 z_t
\]
\[
= \gamma \left\{ -\theta \log E_Y \left[ \exp \left\{ -\frac{(1 - \pi) W_A(Y_{t+1}) + \pi \bar{W}(Y_{t+1}, 0, 0) + A_1 T_\eta/A_1[\epsilon_{t+1}](Y_{t+1})}{\theta} \right\} | y_t \right] \right\} + \gamma(A_0 + A_1 \rho_0) + (1 + \gamma A_1 \rho_1) z_t.
\]
Therefore, the same solution for $A_0$ and $A_1$ holds for $W_A$. Hence,
\[
\bar{W}(y_t, B_t, b_t) = \max_{b_{t+1}} G(b_t, b_{t+1}; B_{t+1}, y_t)
\]
\[
= \gamma \log E_Y \left[ \exp \left\{ -\frac{T_\eta(1 - \gamma \rho_1)[\epsilon_{t+1}](Y_{t+1}, y_t) + \bar{W}(Y_{t+1}, B_{t+1}, b_{t+1})}{\theta} \right\} | y_t \right].
\]

The FONC and envelope conditions for $b_{t+1}$ (assuming interior solution) yield
\[
q^o(y_t, B_{t+1}) = \gamma E_Y \left[ \frac{\exp \left\{ -\frac{T_\eta(1 - \gamma \rho_1)[\epsilon_{t+1}](Y_{t+1}, y_t) + \bar{W}(Y_{t+1}, B_{t+1}, b_{t+1})}{\theta} \right\}}{\exp \left\{ -\frac{T_\eta(1 - \gamma \rho_1)[\epsilon_{t+1}](Y_{t+1}, y_t) + \bar{W}(Y_{t+1}, B_{t+1}, b_{t+1})}{\theta} \right\} | y_t \right] \left| y_t \right],
\]
where $\gamma (w', B') \equiv \left( \lambda + (1 - \lambda)(\psi + q^o(y', B^o(w', B'))) \right)$.

\[\square\]

**Proof of Theorem 4.2.** To show the desired result, it suffices to show that (a) given prices $q$, $W_R(y_t, B_t, b_t, d_t, \xi_t) = W_R(y_t, B_t, b_t) + \bar{U}(d_t, \xi_t)$ and $W_A(y_t, d_t, \xi_t) = W_A(y_t) + \bar{U}(d_t, \xi_t)$ solve the lender’s problem and (b) given the previous solution for the lender’s problem, the pricing condition derived from the FONC yields $q$. 

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To show point (a), observe that

\[ W^e_R(y_t, B_t, b_t, d_t, \xi_t) = \max_{\xi_{t+1}, b_{t+1}} \left\{ z + q(y_t, B_{t+1})(b_{t+1} - (1 - \lambda) b_t) - (\lambda + (1 - \lambda) \psi) b_t + p^e(d_t) \xi_{t+1} - d_t \xi_t \\
+ \min_{m,n} \gamma \left\{ \theta E[m](y_t) + \eta E_Y[m(Y_{t+1})E[n](d_t)]y_t \right\} \\
+ E_Y D[m(Y_{t+1})n(D_{t+1})W_e(Y_{t+1}, B_{t+1}, b_{t+1}, D_{t+1}, \xi_{t+1})][y_t, d_t] \right\}, \tag{33} \]

and

\[ W^e_A(y_t, d_t, \xi_t) = \max_{\xi_{t+1}} \left\{ z + p^e(d_t) \xi_{t+1} - d_t \xi_t + \min_{m,n} \gamma \left\{ \theta E[m](y_t) + \eta E_Y[m(Y_{t+1})E[n](d_t)]y_t \right\} \\
+ E_Y D[m(Y_{t+1})n(D_{t+1})((1 - \pi)W^e_A(Y_{t+1}, D_{t+1}, \xi_{t+1}) + \pi W^e_e(Y_{t+1}, 0, 0, D_{t+1}, \xi_{t+1}))][y_t, d_t] \right\}. \]

Suppose now that \( W^e_R(y_t, B_t, b_t, d_t, \xi_t) = W^e_R(y_t, B_t, b_t) + \mathcal{O}(d_t, \xi_t) \) and \( W^e_A(y_t, d_t, \xi_t) = W^e_A(y_t) + \mathcal{O}(d_t, \xi_t) \) and that \( \delta^e = \delta \) and \( B^e = B \). First, observe that under this assumption

\[ W^e(y_t, B_t, b_t, d_t, \xi_t) = \delta(y_t, B_t) W^e_R(y_t, B_t, b_t) + (1 - \delta(y_t, B_t)) W^e_A(y_t) + \mathcal{O}(d_t, \xi_t) \]

By imposing this in the RHS of the display (33), it follows that

\[ W^e_R(y_t, B_t, b_t, d_t, \xi_t) = \max_{\xi_{t+1}, b_{t+1}} \left\{ z + q(y_t, B_{t+1})(b_{t+1} - (1 - \lambda) b_t) - (\lambda + (1 - \lambda) \psi) b_t + p^e(d_t) \xi_{t+1} - d_t \xi_t \\
+ \min_{m,n} \gamma \left\{ \theta E[m](y_t) + \eta E_Y[m(Y_{t+1})E[n](d_t)]y_t \right\} \\
+ E_Y D[m(Y_{t+1})n(D_{t+1})W(Y_{t+1}, B_{t+1}, b_{t+1}) + \mathcal{O}(D_{t+1}, \xi_{t+1})][y_t, d_t] \right\} \]

where the second equality follows from independence of \( D_t \) and \( Y_t \) and \( E_Y[m(Y_{t+1})y_t] = 1 \) and \( E_D[n(D_{t+1})d_t] = 1 \); the third equality follows from the linearity of the per-period payoff. It is easy to see that the first two summands of the RHS are equal to \( W^e_R(y_t, B_t, b_t) \) because \( W^e_R \) is precisely the solution to the functional equation. Observe that by solving the minimization in the third summand of the RHS, it becomes

\[ \max_{\xi_{t+1}} \left\{ p^e(d_t) \xi_{t+1} - d_t \xi_t - \gamma \eta \log E_D \left[ \exp(-\eta^{-1}\mathcal{U}(D_{t+1}, \xi_{t+1})) \right] d_t \right\}, \]

and by definition equals \( \mathcal{U}(\xi_t, d_t) \). An analogous result follows for \( W^e_A \). This shows point (a).

From the expression above it is easy to see that the \( b_{t+1} \) that maximizes the lender’s problem coincides with the one in the economy without the asset. This shows point (b).
Finally, by computing FONC for $\xi_{t+1}$ it is easy to see that

$$p^f(d_t) = \gamma E_D \left[ D_{t+1} \exp \left\{ -\eta^{-1} \phi(D_{t+1}, \Xi_{t+1}) \right\} \right].$$

\[\square\]

F Economy with Incomplete Information

We consider a slightly different version of the borrower’s problem, and thus of the whole economy, studied in the text. We now assume that in every period $t$, after the state is realized, the borrower receives an i.i.d. preference shock $\nu$, drawn from $P_\nu(\cdot|s)$, where $s$ is defined below, before choosing its action. The preference shock $\nu$ is known to the borrower but not to the lender. The period payoff of the borrower is given by $u + \nu \times (1 - \delta)$. We refer to this new economy with private information, as the perturbed economy.

Throughout this section we changed the notation used in the text to a more succinct one (albeit less standard). This new notation follows closely that in Doraszelski and Escobar (2010) and facilitates the proof of the theoretical results in this section.

Let $s \equiv (B, y, \phi) \in S \equiv B \times Y \times \{0, 1\}$ be the aggregate state, where $B$ is the government debt, $y$ is the endowment and $\phi$ is an indicator variables such that $\phi = 0$ states that the economy is in financial access, and $\phi = 1$ states that the economy is in financial autarky. The law of motion for $\phi$ is given by

$$\Pr\{\phi' = 0 \mid \phi = 1\} = \pi$$

and

$$\Pr\{\phi' = 0 \mid \phi = 0\} = \sigma(1|s) + \sigma(0|s)\pi,$$

where $\sigma(\delta|s)$ is the probability of the borrower to choose $\delta$, given $s$ (see below).

Recall that $\delta \in \{0, 1\}$ is the action of default of the government, $\delta = 1$ means no default and $\delta = 0$ means default. The possible actions of default are state dependent in the following sense $A(s) = \{0, 1\}$ if $\phi = 0$ and $A(s) = \{0\}$ if $\phi = 1$. That is, if the economy is in financial autarky, then the government does not have a choice and ought to remain in financial autarky. We use $B$ to denote the government debt policy function, that is $B : S \times \{0, 1\} \to \mathbb{R}$ such that $B(s; q)$ is the choice of new bond holdings, given the state $s$, a choice of default $\delta = 1$ (there is no need to define $B$ when $\delta = 0$ since the economy is in financial autarky) and a belief about the pricing function.

Let $s_L \equiv (b, s) \in B \times S$ be the state of the lender. We use $b$ to denote the debt policy function for the lenders, that is $b(s_L; \sigma, B, q)$ is the the choice of lender’s debt for “tomorrow”, given the state $s_L$ and beliefs about $\sigma, B, q$.

Our goal is to show that when $P_\nu(\cdot|s)$ converges to a degenerate measure at zero (in a certain sense to be specified below), the solution of the perturbed economy, if it converges, it does so to the solution of the economy without preference shock. Hence we consider a sequence of cdfs $(F^n_\nu(\cdot|s))_{n \in \mathbb{N}}$ which has the following property

**Assumption F.1.** For any $s \in S$ and $n \in \mathbb{N}$, (i) if $\phi = 1$, then $P^n_\nu(v|s) = \delta_0(v)$ and if $\phi = 0$, then
$F^n_\nu(\cdot|s)$ is absolutely continuous with respect to the Lebesgue measure, and

$$\lim_{n \to \infty} \sup_{s \in S} \frac{1}{P^n_\nu(A^n|s)} \int_{A^n} \nu F^n_\nu(d\nu|s) = 0;$$

for any sequence of Borel measurable sets $A^n$.

This assumption is analogous to that in Doraszelski and Escobar (2010). The only caveat is that, due to the characteristics of our problem, we impose that $\nu \equiv 0$ whenever the economy is in financial autarky. Is only when the government actually chooses to default that the shock is present. This particular choice stems from the fact that our interest of the shock is merely as a numerical device to “smooth” the default decision.

We use $\{(W^n, V^n), (b^n, B^n), \sigma^n, q^n\}$ to denote quantities corresponding to the perturbed economy with preference shock distribution $F^n_\nu$, and we include a superscript $*$ to denote equilibrium quantities.

### F.1 Borrower’s Problem

We first present the transition probabilities of the state for this new formulation and then present the Bellman equation for the borrower. For simplicity, we assume that $Y$ has a pdf which we denote as $f$.

#### F.1.1 Transition state probabilities

Let $C \equiv C_B \times C_Y \times C_\phi$ where $C_B$ and $C_Y$ are Borel measurable subsets of $Y$ and $B$, and $C_\phi \in 2^{\{0,1\}}$. For almost any $\bar{s} \equiv (\bar{B}, \bar{y}, \bar{\phi}) \in S$ we compute the conditional probability of $C$, given $\bar{s}$ and given the choices $\delta$ and $\tilde{B}$, that is, $Q\left(s' \in C \mid s = \bar{s}; \sigma, \tilde{B}\right)$. For the case $\bar{\phi} = 0$,

$$Q\left(s' \in C \mid s = \bar{s}; \sigma, \tilde{B}\right) = 1\{\tilde{B} \in C_B\}(1\{0 \in C_\phi\}\sigma(0|\bar{s}) + \pi 1\{0 \in C_\phi\} + (1 - \pi) 1\{1 \in C_\phi\}) \sigma(1|\bar{s})) \times \int_{y' \in C_Y} f(y'|\bar{y})dy'.$$

For the case $\bar{\phi} = 1$,

$$Q\left(s' \in C \mid s = \bar{s}; \sigma, \tilde{B}\right) = 1\{\tilde{B} \in C_B\}(1\{0 \in C_\phi\}\pi + 1\{1 \in C_\phi\})(1 - \pi)) \int_{y' \in C_Y} f(y'|\bar{y})dy'.$$

#### F.1.2 Recursive Formulation

The borrower’s Bellman equation can be written succinctly as

$$V^n(s; q) = \int_{R} \left( \max_{\delta \in A(s)} V^n(\delta, s; q) + \nu(1 - \delta) \right) F^n_\nu(d\nu|s),$$

where $\nu(1 - \delta)$ implies that the preference shock only occurs if the borrower is actually choosing default, and not if it is already in financial autarky; also

$$V^n(\delta, s; q) \equiv U(s, \delta, B^n(s; q); q) + \beta \int_S V^n(s'; q)Q(ds' \mid s; \delta, B^n(s; q)),$$
and

\[ B^n(s; q) \in \arg \max_{B' \in \mathbb{B}(\delta)} \left\{ U(s, \delta, B'; q) + \beta \int_\mathcal{S} V^n(s'; q)Q(ds' \mid s; \delta, B') \right\}; \]

where \( \mathbb{B}(\delta) \equiv \mathbb{B} + (1 - \delta)\{0\} \). Finally, for all \( s \in \mathcal{S} \),

\[ U(s, \delta, B'; q) \equiv (1 - \delta)u(y - \phi(y)) + \delta u(y - q(y, B')B' + B). \]

Let \( \delta(s, \nu) \) be the best response of the borrower in state \( s \), and shock \( \nu \), i.e.

\[ \delta^n(s, \nu) = \begin{cases} 1 & \text{if } V^n(1, s; q) > V^n(0, s; q) + \nu \\ 0 & \text{if } V^n(1, s; q) < V^n(0, s; q) + \nu \end{cases} \]

We note that, since \( \nu \) is assumed to have absolutely continuous distribution; the case \( V^n(1, s; q) = V^n(0, s; q) + \nu \) occurs with probability zero.

Note that

\[ \sigma^n(a \mid s) \equiv \int_{\nu: \delta^n(s, \nu) = a} F^n(\nu \mid s). \]

We let \( U(s, \sigma, B'; q) \equiv \sigma(1 \mid s)U(s, 1, B'; q) + \sigma(0 \mid s)U(s, 0, B'; q) \). The next lemma characterizes the properties of \( V^n \).

**Lemma F.1.** Suppose Assumption F.2 holds and \( q(y, \cdot) \) is continuous. Then (1) For each \( B' \in \mathbb{B} \), \( U(\cdot, \sigma, B'; q') \) is an uniformly bounded and continuous function and (2) \( V^n(\cdot; q) \) is an uniformly bounded and continuous function.

**Proof.** The proof follows from standard arguments; see Stokey and Robert E. Lucas (1989). \( \square \)

For example, if \( F^n(\nu) \) is given by the logistic distribution, i.e., \( \nu \rightarrow F^n(\nu) = \frac{1}{1 + \exp\{-h_n \nu\}} \) (\( h_n \) controls the variance of \( F^n(\nu) \)), equation (34) has a closed form solution, given by

\[
V^n(s; q) = \int_{\mathbb{R}} \max \left\{ 0, V^n(1, s; q) - V^n(0, s; q) + \nu \right\} F^n(\nu) \, d\nu + V^n(0, s; q)
\]

\[
= \int_{-\Delta V^n(s, q)}^{\infty} \{\Delta V^n(s; q) + \nu\} F^n(\nu) \, d\nu + V^n(0, s; q)
\]

\[
= -\nu(1 - F^n(\nu)) \bigg|_{\nu=0}^{\nu=\infty} + \int_0^{\infty} 1 - F^n(\nu - \Delta V^n(s; q)) \, d\nu
\]

\[
= h_n \log \left( 1 + \exp \left\{ h_n^{-1} \Delta V^n(s; q) \right\} \right).
\]

where \( \Delta V^n(s; q) = V^n(1, s; q) - V^n(0, s; q) \). The third line follows by integration by parts. The fourth line follows from the properties of the logistic distribution.

**F.2 The Lender’s Problem**

For the sake of presentation, and given that we introduced new notation, we present the lender’s problem. This part, as opposed to the Borrower’s, is just a reformulation of what we have in the text.
The conditional probability is given by $Q_L (\cdot \mid s_L = \bar{s}_L; \tilde{b}, \sigma, B)$ where $B$ and $\sigma$ are beliefs of the lender about the policy functions of the government; the functional form is analogous to that of $Q$ and thus, it will not be derived again.

Let

$$W(s_L; \sigma, B, q) = \left\{ \mathcal{U}(s_L, b(s_L; \sigma, B, q); \sigma, B) + \gamma T^\theta[W](s_L, b(s_L; \sigma, B, q); \sigma, B) \right\},$$

where

$$b(s_L; \sigma, B, q) \in \arg \max_{b' \in B(\phi)} \left\{ \mathcal{U}(s_L, b'; \sigma, B) + \gamma T^\theta[W](s_L, b'; \sigma, B) \right\},$$

and $T^\theta$ is given by

$$W \mapsto T^\theta[W](s_L; \tilde{b}, \sigma, B) = -\theta \log \left( \int_{\mathbb{Y}} \exp \left\{ -\frac{W(y', s_L; \tilde{b}, \sigma, B)}{\theta} \right\} \right),$$

where

$$W(y', s_L; \tilde{b}, \sigma, B) = \int_{S} W(s'_L) Q_L (ds'_L \mid y', s_L; \tilde{b}, \sigma, B),$$

(the measure used for integration is the measure $Q_L$ conditional on $y'$; abusing notation we still denote it as $Q_L$).

The problem of the lender is not indexed by $n$ directly, it will depend on $n$ only indirectly through its (correct) beliefs of the borrowers policy functions and prices. For convenience, however, we use $W_n(s_L; q^n)$ to denote $W(s_L; \sigma^n, B^n, q^n)$.

The next assumption ensures that positive consumption for the borrower is always feasible.

**Lemma F.2.** Suppose $q(y, \cdot)$ is continuous. Then $W(\cdot; q)$ is a bounded and continuous function.

**Proof.** The proof follows from standard arguments; see Stokey and Robert E. Lucas (1989). \qed

### F.3 Convergence of Economy with Incomplete Information

The definition of equilibrium in the economy with incomplete information is analogous to the one in the text and therefore, will be omitted. We denote the equilibrium of the perturbed economy as \{(W^{*\ast, n}, V^{*\ast}), (b^{*\ast, n}, B^{*\ast, n}), \sigma^{*\ast, n}, q^{*\ast, n}\}.

We note that, if $q$ is an equilibrium price (given the other equilibrium quantities), then

$$q(s, B') = \gamma \int_{y'\delta(B', y', 0)=1} \mathcal{m}_R(y'; y, B') f(y'|y)dy' \leq \gamma.$$  \hfill (35)

Hence $(s, B') \mapsto q(s, B') \in [0, \gamma]$.

The next assumption ensures that positive consumption for the borrower is always feasible.
**Assumption F.2.** There exists a $C_1 > 0$, such that, for all $(y, B) \in \mathbb{Y} \times \mathbb{B}$,

$$\sup_{B' \in \mathbb{B}} (y + B) - \gamma B' \geq C_1.$$ 

We now establish the main result of this section: as shocks vanishes, and if $(b^{*, n}, B^{*, n}, \sigma^{*, n}, q^{*, n})$ converge, then the equilibrium of the perturbed game converges to an equilibrium in economy without noise, i.e., the economy described in the text.

**Theorem F.1.** Suppose Assumptions F.1 and F.2 hold, and suppose $q^{*, n}(y, \cdot)$ is continuous for each $n$. If $\lim_{n \to \infty} ||\sigma^{*, n} - \sigma^*||_{L^\infty(\{0, 1\} \times \mathbb{S})} = 0$, $\lim_{n \to \infty} ||B^{*, n} - B^*||_{L^\infty(\mathbb{S})} = 0$ and $\lim_{n \to \infty} ||q^{*, n} - q^*||_{L^\infty(\mathbb{S})} = 0$, then $(\sigma^*, B^*)$ are an equilibrium debt policy functions and default strategy and $q^*$ is a continuous equilibrium price function of the economy without noise.$^7$

**Proof.** We divide the proof into several steps. Throughout the proof $supp(\sigma(\cdot|s))$ is the set of all $d$ such that $\sigma(d|s) > 0$.

**Step 1.** We show that if $\delta \in supp(\sigma(\cdot|s))$ then $\delta \in \arg\max (V^{*, n}(\delta, s; q))$. Since $\lim_{n \to \infty} ||\sigma^{*, n} - \sigma^*||_{L^\infty(\{0, 1\} \times \mathbb{S})} = 0$, then there exists a $N$, such that $supp(\sigma^*(\cdot|s)) \subseteq supp(\sigma^{*, n}(\cdot|s))$ for all $n \geq N$. For any $\delta \in supp(\sigma^*(\cdot|s)) = supp(g^*(\cdot|s))$, since $F^n_\nu$ is absolutely continuous with respect to the Lebesgue measure, there exists a set $D^\prime_s \subseteq \mathbb{R}$ (of positive measure) such that

$$V^{*, n}(\delta, s; q) - V^{*, n}(\delta', s; q) > \nu(1 - \delta') - \nu(1 - \delta),$$

for all $\delta' \neq \delta$ and all $\nu \in D^\prime_s$. Integrating at both sides it follows

$$V^{*, n}(\delta, s; q) - V^{*, n}(\delta', s; q) > \frac{1}{P^n_\nu(D^\prime_s)} \int_{\mathbb{R}} \{\nu(1 - \delta') - \nu(1 - \delta)\} F^n_\nu(d\nu|s).$$

As $n \to \infty$, by Assumption F.1, the RHS converges to zero, and by Theorem F.3, the LHS converges to $V^*(\delta, s; q) - V^*(\delta', s; q)$, and the result thus follows.

**Step 2.** We show that, for any $s$ and $d$, $B^*(s)$ is optimal for the borrower, given $\sigma^*$ and $q^*$. It suffices to show this for $\delta = 1$ (for $\delta = 0$ there is no choice of debt). We note that since $q^{*, n}$ is continuous and converges uniformly to $q^*$ over a compact set; $q^*$ is also continuous.

We know that

$$U(s, 0, B^*(s; q^{*, n}); q^{*, n}) + \beta \int_{\mathbb{S}} V^{*, n}(s'; q^{*, n})Q(ds'|s; 0, B^*(s; q^{*, n})) \geq U(s, 0, B'; q^{*, n})$$

$$+ \beta \int_{\mathbb{S}} V^{*, n}(s'; q^{*, n})Q(ds'|s; 0, B'),$$

for all $B' \in \mathbb{B}$.

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$^7$The norm $||\cdot||_{L^\infty(X)}$ is the standard supremum norm over functions that map $X$ to $\mathbb{R}$.  

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Note that
\[
\sup_{B'} |U(s, 0, B'; q^{*,n}) - U(s, 0, B'; q^*)| = \sup_{B'} |u(y + B - q^{*,n}(y, B')B') - u(y + B - q^*(y, B')B')|
\]
\[
\leq \sup_{B'} \left| \frac{du(y + B - q^{*,n}(y, B')B')}{dc} \right| |B'\{q^{*,n}(y, B') - q^*(y, B')\}|
\]
\[
= O(||q^{*,n} - q^*||_{L^\infty(S)}) = o(1); \quad (38)
\]
where \( \bar{q}^{*,n}(y, B') \) is a mid-point between \( q^{*,n}(y, B') \) and \( q^*(y, B') \). The second line follows from differentiability of \( u \); the third line follows from the fact that \( B \) is compact (and hence bounded) and \( \sup_{B'} \{y + B - \bar{q}^{*,n}(y, B')B'\} \geq C_1 \) (by Assumption F.2) and \( c \mapsto \frac{du(c)}{dc} \) is bounded at \( C_1 \). Also, by Lemma F.3, we know that, for any \( s \),
\[
\sup_{B'} \left| \int_S V^*(s'; q^{*,n})Q(ds' \mid s; 0, B') - \int_S V^*(s'; q^*)Q(ds' \mid s; 0, B') \right| = o(1). \quad (39)
\]
Hence (we omit the inputs in the policy function to ease the notational burden)
\[
U(s, 0, B^*; q^*) + \beta \int_S V^*(s'; q^*)Q(ds' \mid s; 0, B^*)
\]
\[
\geq U(s, 0, B^*; q^{*,n}) + \beta \int_S V^{*,n}(s'; q^{*,n})Q(ds' \mid s; 0, B^{*,n}) - A_n(s)
\]
\[
\geq U(s, 0, B'; q^{*,n}) + \beta \int_S V^{*,n}(s'; q^{*,n})Q(ds' \mid s; 0, B') - A_n(s)
\]
\[
\geq U(s, 0, B'; q^*) + \beta \int_S V^*(s'; q^*)Q(ds' \mid s; 0, B') - 2A_n(s),
\]
where
\[
A_n(s) = \sup_{B'} \left| U(s, 0, B'; q^{*,n}) + \beta \int_S V^{*,n}(s'; q^{*,n})Q(ds' \mid s; 0, B')
\]
\[
\quad - \left\{ U(s, 0, B'; q^*) + \beta \int_S V^*(s'; q^*)Q(ds' \mid s; 0, B') \right\},
\]
and the second inequality follows from equation (36). This equation holds for any \( n \), in particular as \( n \to \infty \). By equation (39), \( A_n(s) = o(1) \), and the result thus follows.

**Step 3.** We show that, for any \( s, B' \mapsto q^*(s, B') \) is an equilibrium price schedule that supports \( \sigma^* \) and \( B^* \). For this, it suffices to verify that
\[
q^*(s, B') = \gamma \int_Y \sigma(1|B', y', 0) \exp\left\{ -\theta^{-1}W^*(B', B', y', 0; q^*) \right\} \int_Y \exp\left\{ -\theta^{-1}W^*(B', B', y', 0; q^*) \right\} f(y'|y)dy'.
\]
To establish this, we note that for all \((s, B')\),
\[
q^*(s, B') - \frac{\gamma}{\calY} \sigma^* (1 | B', y', 0) \exp \left\{ -\theta^{-1}W^*/(B', B', y', 0; q^*) \right\} \frac{f(y') \, dy'}{\int_{\calY} \exp \left\{ -\theta^{-1}W^*/(B', B', y', 0; q^*) \right\}} \leq q^*(s, B') - q^* n (s, B')
\]
\[
+ \frac{\gamma}{\calY} \left( \sigma^* n (1 | B', y', 0) \exp \left\{ -\theta^{-1}W^*/(B', B', y', 0; q^* n) \right\} \right) \frac{f(y') \, dy'}{\int_{\calY} \exp \left\{ -\theta^{-1}W^*/(B', B', y', 0; q^* n) \right\}}
\]
\[
- \sigma^* (1 | B', y', 0) \exp \left\{ -\theta^{-1}W^*/(B', B', y', 0; q^*) \right\} \frac{f(y') \, dy'}{\int_{\calY} \exp \left\{ -\theta^{-1}W^*/(B', B', y', 0; q^*) \right\}}.
\]

The first term in the RHS vanishes as \(n \to \infty\) by assumption. Since \(\lim_{n \to \infty} \|\sigma^* n - \sigma^*\|_{L^\infty(0, 1) \times \mathcal{S}} = 0\) and \(W \mapsto \exp \left\{ -\theta^{-1}W \right\}\) is continuous (under the \(\| \cdot \|_{L^\infty(\mathcal{S})}\) norm) it suffices to show that \(\|W^*/(\cdot; q^* n) - W^*/(\cdot; q^*)\|_{L^\infty(\mathcal{S})} = o(1)\). This follows from Lemma F.3 and similar calculations to those done in equations (38) and (39).

In order to establish Lemma F.3 below. It is convenient to introduce some notation. Let \(\mathbb{L}_{\sigma, B} : L^\infty(\mathcal{S}) \to L^\infty(\mathcal{S})\), such that
\[
\mathbb{L}_{\sigma, B} [v] (s) \equiv \int_{\mathcal{S}} v(s') Q(ds' | s; \sigma, B(s; q)).
\]

It is easy to see that \(\mathbb{L}_{\sigma, B}\) is a linear operator. Let \(\mathcal{L}(L^\infty(\mathcal{S}))\) be the class of linear operators that map \(L^\infty(\mathcal{S})\) onto \(L^\infty(\mathcal{S})\). Endowed with the operator norm, i.e., \(\|L_{\sigma, B}\| \equiv \sup_{v \in L^\infty(\mathcal{S})} \|\mathbb{L}_{\sigma, B} [v]\|_{L^\infty(\mathcal{S})} / \|v\|_{L^\infty(\mathcal{S})}\), the class \(\mathcal{L}(L^\infty(\mathcal{S}))\) is a Banach Algebra, see Lax (2002). This implies that algebraic properties enjoyed by algebras, such as multiplication of elements, can be applied to \(\mathbb{L}_{\sigma, B}\). In particular, following analogous steps to those in Doraszelski and Escobar (2010) p. 381, it follows that
\[
V^n (s; q) = U(s, \sigma, B(s; q); q) + \sigma (0 | s) \int_{\nu; \delta^n (s, \nu) = 0} \nu F^n (d\nu | s) + \beta \int_{\mathcal{S}} V^n (s'; q) Q(ds' | s; \sigma, B(s; q))
\]
\[
\equiv U(s, \sigma, B(s; q); q) + e^n (s; \sigma) + \beta \int_{\mathcal{S}} V^n (s'; q) Q(ds' | s; \sigma, B(s; q)).
\]

Can be cast as
\[
V^n (s; q) = \left( \sum_{t=0}^{\infty} \beta^t \mathbb{L}_{\sigma, B}^t \right) \left[ U(\cdot, \sigma, B(s; q); q) + e^n (\cdot; \sigma) \right](s);
\]
where \(\mathbb{L}_{\sigma, B}^t\) is the composition of the operator applied \(t\) times. Equivalently (see Lemma F.4)
\[
V^n (s; q) = (I - \beta \mathbb{L}_{\sigma, B})^{-1} \left[ U(\cdot, \sigma, B(s; q); q) + e^n (\cdot; \sigma) \right](s).
\]

In particular,
\[
V^{*, n} (s; q^* n) = (I - \beta \mathbb{L}_{\sigma^* n, B^* n})^{-1} \left[ U(\cdot, \sigma^* n, B^* n (s; q^* n); q^* n) + e^n (\cdot; \sigma^* n) \right](s).
\]
And,
\[ V^*(s; q^*) = (I - \beta \mathbb{1}_{\sigma^*,B^*})^{-1} [U(\cdot, \sigma^*, B^*(s; q^*); q^*)](s). \]  
(41)

Let \( C(S) \) be the class of continuous functions in \( L^\infty(S) \).

**Lemma F.3.** Suppose Assumption F.2 holds, \( q^*(y, \cdot) \) is continuous, and suppose \( \lim_{n \to \infty} ||\sigma^*,n - \sigma^*||_{L^\infty((0,1) \times S)} = 0 \), \( \lim_{n \to \infty} ||B^{*,n} - B^*||_{L^\infty(S)} = 0 \) and \( \lim_{n \to \infty} ||q^{*,n} - q^*||_{L^\infty(S)} = 0 \). Then, for any \( s \in S \),
\[ \lim_{n \to \infty} \sup_{B'} \left| \int_S V^{*,n}(s'; q^{*,n})Q(ds' \mid s; 0, B') - \int_S V^*(s'; q^*)Q(ds' \mid s; 0, B') \right| = o(1). \]  
(42)

And,
\[ \lim_{n \to \infty} \sup_{B'} \left| \int_S W^{*,n}(s'; q^{*,n})Q_L(ds' \mid s; B', 0, B') - \int_S W^*(s'; q^*)Q_L(ds' \mid s; B', 0, B') \right| = o(1). \]  
(43)

**Proof of Lemma F.3.** The proof for equation (43) is completely analogous to that of equation (42); hence we only show the latter.

We note that
\[ \int_S V^{*,n}(s'; q^{*,n})Q(ds' \mid s; 0, B') = \int_S V^{*,n}(B', y', 0; q^{*,n}) f(y' \mid y)dy', \]
(and the same for \( V^* \)). Hence, since \( V^{*,n} \) (and \( V^* \)) belong to \( L^\infty(S) \) (see Lemma F.1), by the Dominated Convergence Theorem, it suffices to show that
\[ \lim_{n \to \infty} \sup_{B'} |V^{*,n}(B', y', 0; q^{*,n}) - V^*(B', y', 0; q^*)| = 0, \]
pointwise on \( y' \in \mathbb{Y} \). By equations (40)-(41), we can cast \( V^{*,n} \) and \( V^* \) in terms of \( (I - \beta \mathbb{1}_{\sigma,B})^{-1} \) and thus, it suffices to show
\[ \lim_{n \to \infty} \sup_{B} |M_{\sigma^{*,n}, B^{*,n}}[e^n(\cdot; \sigma^{*,n})](s)| = 0, \]  
(44)

\[ \lim_{n \to \infty} \sup_{B} \left| \{M_{\sigma^{*,n}, B^{*,n}} - M_{\sigma^{*}, B^*}\}[U(\cdot, \sigma^*, B^*(s; q^*); q^*)](s) \right| = 0, \]  
(45)

and
\[ \lim_{n \to \infty} \sup_{B} |M_{\sigma^{*}, B^*}[U(\cdot, \sigma^*, B^*(s; q^{*,n}); q^{*,n})] - U(\cdot, \sigma^*, B^*(s; q^*); q^*)](s)| = 0, \]  
(46)

where \( M_{\sigma,B} \equiv (I - \beta \mathbb{1}_{\sigma,B})^{-1} \).

Regarding equation (44), note that
\[ \sup_{B} |M_{\sigma^{*,n}, B^{*,n}}[e^n(\cdot; \sigma^{*,n})](s)| \leq \|M_{\sigma^{*,n}, B^{*,n}}\| \times \|e^n(\cdot; \sigma^{*,n})\|_{L^\infty(S)}. \]

The operator norm of \( M_{\sigma^{*,n}, B^{*,n}} \), \( \|M_{\sigma^{*,n}, B^{*,n}}\| \leq \sum_{i} \beta^i \|\mathbb{I}_{\sigma^{*,n}, B^{*,n}}\|^i \leq (1 - \beta)^{-1} \). Hence, it suffices
to show that \(|e^n(\cdot; \sigma^{*,n})|_{L^\infty(S)} = o(1)|. By definition of \(e^n\),

\[ ||e^n(\cdot; \sigma^{*,n})||_{L^\infty(S)} \leq 2\int_{\mathbb{R}} \nu F^n_{\nu}(dv|s)||_{L^\infty(S)}. \]

From Assumption F.1, the RHS is of order \(o(1)|.

Regarding equation (45), since \(U(\cdot, \sigma^{*}, B^*(s; q^{*}); q^{*})\) is uniformly bounded and continuous (see Lemma F.1), the desired result follows from Lemma F.4(2).

Regarding equation (46), note that

\[ \sup_B |M_{\sigma^{*,n}, B^{*,n}}[U(\cdot, \sigma^{*,n}, B^{*,n}(s; q^{n,*}); q^{n,*}) - U(\cdot, \sigma^{*}, B^*(s; q^{*}); q^{*})](s)| \leq ||M_{\sigma^{*,n}, B^{*,n}}|| \times ||U(\cdot, \sigma^{*,n}, B^{*,n}(s; q^{n,*}); q^{n,*}) - U(\cdot, \sigma^{*}, B^*(s; q^{*}); q^{*})||_{L^\infty(S)}. \]

Since \(||M_{\sigma^{*,n}, B^{*,n}}|| \leq (1 - \beta)\) (see above), it suffices to show

\[ ||U(\cdot, \sigma^{*,n}, B^{*,n}(s; q^{n,*}); q^{n,*}) - U(\cdot, \sigma^{*}, B^*(s; q^{*}); q^{*})||_{L^\infty(S)} = o(1). \]

By similar algebra to that in equation (38), the previous expression can be bounded above by

\[ \sup_{y,B,B'} \left| \frac{d\alpha(y + \tilde{B} - \tilde{q}^{*,n}(y, B') B')}{dc} \right| \left\{ \gamma \|B^* - B^{*,n}\|_{L^\infty(S)} + \tilde{B} \|q^* - q^{n,*}\|_{L^\infty(S)} \right\}; \]

since \(q^{n,*} \leq \gamma\) (see equation (35)) and \(\tilde{B} < \infty\) (because \(B\) is bounded). Under Assumption F.2 and hence, the previous expression is of order \(O\left( \max\{\|q^* - q^{n,*}\|_{L^\infty(S)}, \|B^* - B^{*,n}\|_{L^\infty(S)} \right) \)).

Let \(C(S)\) be the space of continuous and uniformly bounded functions that map \(S\) onto \(\mathbb{R}\).

**Lemma F.4.** For any \((\sigma, B)\): (1)

\[ \left( \sum_{t=0}^{\infty} \beta^t L_{\sigma,B}^t \right) = (I - \beta L_{\sigma,B})^{-1} \]

exists as an element of \(L(L^\infty(S));\) (2) For any \(g \in C(S)\) and \(s\), \((\sigma, B) \mapsto (I - \beta L_{\sigma,B})^{-1} [g](s)\) is continuous under the \(L^\infty(\{0, 1\} \times S) \times L^\infty(S \times \{0, 1\}) \) \(\| \cdot \|\) norms.

**Proof of Lemma F.4.** (1) We note that \(||L_{\sigma,B}|| \leq 1\), hence \(||\sum_{t=0}^{T} \beta^t L_{\sigma,B}^t \| \leq \sum_{t=0}^{T} \beta^t ||L_{\sigma,B}|| \leq (1 - \beta)^{-1}\). It is easy to see that this implies that the partial sums, \((\sum_{t=0}^{T} \beta^t L_{\sigma,B}^t)_{T}\), are a Cauchy sequence, and since \(L(L^\infty(S))\) is complete, it converges (to \(S\)). Pre-multiplying by \(\beta L_{\sigma,B}\), it follows

\[ \beta L_{\sigma,B} S = \sum_{t=1}^{\infty} \beta^t L_{\sigma,B}^t = S - I. \]

Hence \(I = S(I - \beta L_{\sigma,B})\). Post-multiplying by \(\beta L_{\sigma,B}\) an analogous result follows and thus the desired result is proven.

(2) We want to show that, for any \(\epsilon > 0\), there exists \(\eta > 0\) such that

\[ \sup_B \left( \sum_{t=0}^{\infty} \beta^t L_{\sigma,B}^t - \sum_{t=0}^{\infty} \beta^t L_{\sigma,B}^t \right) [g](s) < \epsilon, \quad (47) \]
for all \((\tilde{\sigma}, \tilde{B})\) such that \(\|\tilde{\sigma} - \sigma\|_{L^\infty([0,1] \times \mathbb{S})} + \|\tilde{B} - B\|_{L^\infty(\mathbb{S})} \leq \eta\).

The LHS of equation (47), is bounded above by

\[
\sup_B \left( \frac{1}{T} \sum_{t=0}^{T-1} \beta_t L_{\sigma,B}^t - \sum_{t=0}^{T-1} \beta_t L_{\sigma,B}^t \right) g(s) + \sum_{t=T+1}^{\infty} \beta_t L_{\sigma,B}^t - \sum_{t=T+1}^{\infty} \beta_t L_{\sigma,B}^t \right) ||g||_{L^\infty(\mathbb{S})}.
\]

Since \(||g||_{L^\infty(\mathbb{S})} < \infty\) and \(||L_{\sigma,B}|| \leq 1\) for all \((\sigma, B)\), it follows there exists a \(T(\epsilon)\) such that the second summand is less than \(0.5\epsilon\). Since \(T(\epsilon) < \infty\), to show that the first summand is less than \(0.5\epsilon\), it suffices to show that there exists a \(\eta > 0\), such that

\[
\sup_B \left| \left( \frac{1}{T} \sum_{t=0}^{T-1} \beta_t L_{\sigma,B}^t - \sum_{t=0}^{T-1} \beta_t L_{\sigma,B}^t \right) g(s) \right| < \epsilon, \tag{48}
\]

for all \((\tilde{\sigma}, \tilde{B})\) such that \(||\tilde{\sigma} - \sigma\|_{L^\infty([0,1] \times \mathbb{S})} + ||\tilde{B} - B||_{L^\infty(\mathbb{S})} \leq \eta\). We note that the \(\delta\) will depend on \(T(\epsilon)\) too.

We can cast the RHS of equation (48) as

\[
\sup_B \left| \int_{\mathbb{Y}} \left\{ \sum_{\delta \in (0,1)} \tilde{\sigma}(\delta|s)g(\tilde{B}(\cdot), y', \delta) - \sigma(\sigma|s)g(B(\cdot), y', \delta) \right\} f(y')dy' \right|.
\]

Since \(||\tilde{\sigma} - \sigma\|_{L^\infty([0,1] \times \mathbb{S})} < \delta\) and \(g \in L^\infty(\mathbb{S})\) —by the Dominated convergence theorem—it suffices to show, for all \((\delta, y)\) and \(\epsilon > 0\), there exists a \(\eta > 0\) (it may depend on \((\delta, y)\)) such that

\[
\sup_B \left| g(\tilde{B}(\cdot), y, \delta) - g(B(\cdot), y, \delta) \right| < \epsilon,
\]

or all \((\tilde{\sigma}, \tilde{B})\) such that \(||\tilde{\sigma} - \sigma\|_{L^\infty([0,1] \times \mathbb{S})} + ||\tilde{B} - B||_{L^\infty(\mathbb{S})} \leq \eta\).

To show this, we note that \(B' \mapsto g(B', y, \delta)\) is continuous. Hence, for every \(B\), there exists an \(\eta_B(\epsilon, y, \delta)\) such that

\[
\left| g(\tilde{B}(B, y), y, \delta) - g(B(B, y), y, \delta) \right| < \epsilon
\]

for all \(|\tilde{B}(B, y) - B(B, y)| < \eta_B(\epsilon, y, \delta)\). By choosing \(||\tilde{B} - B||_{L^\infty(\mathbb{S})} < \eta(\epsilon, y, \delta)\), the previous equation ensures that

\[
\sup_B \left| g(\tilde{B}(B, y), y, \delta) - g(B(B, y), y, \delta) \right| < \epsilon.
\]

Letting \(\eta \equiv \eta(\epsilon, y, \delta)\) ensures the desired result. \(\square\)